

INTRINSICALLY NONPERTURBATIVE QCD

I. A PURE DYNAMICAL THEORY OF GLUON CONFINEMENT

V. Gogohia*

HAS, CRIP, RMKI, Depart. Theor. Phys., Budapest 114, P.O.B. 49, H-1525, Hungary

(Dated: February 1, 2008)

We establish exactly and uniquely the infrared structure of the full gluon propagator in QCD, not solving explicitly the corresponding dynamical equation of motion. By construction, this structure is an infinite sum over all severe (i.e., more singular than $1/q^2$) infrared singularities. It reflects the zero momentum modes enhancement effect in the true QCD vacuum. Its existence exhibits a characteristic mass (the so-called mass gap), which is responsible for the scale of nonperturbative dynamics in the QCD ground state. The theory of distributions, complemented by the dimensional regularization method, allows one to put severe infrared singularities under firm mathematical control. By an infrared renormalization of a mass gap only, the infrared structure of the full gluon propagator is exactly reduced to the simplest severe infrared singularity, the famous $(q^2)^{-2}$. So, the smooth in the infrared limit the full gluon propagator is to be ruled out. Collective motion of all the purely transverse *virtual* gluon field configurations with low-frequency components/large scale amplitudes is solely responsible for the color confinement phenomenon within our approach. At the microscopic, dynamical level these field configurations are saturated by the nonlinear fundamental four-gluon interaction. It just makes the full gluon propagator inevitably so singular in the infrared. The amplitudes of all the purely transverse severely singular *actual* gluon field configurations are totally suppressed, leading thus to the confinement of gluons. We formulate the gluon confinement criterion in a manifestly gauge-invariant way, taking into account the distribution nature of severe infrared singularities.

PACS numbers: 11.15.Tk, 12.38.Lg

I. INTRODUCTION

Quantum Chromodynamics (QCD) [1] is widely accepted as a realistic, quantum field gauge theory of strong interactions not only at the fundamental (microscopic) quark-gluon level, but at the hadronic (macroscopic) level as well. This means that, in principle, it should describe the properties of experimentally observed hadrons in terms of experimentally never seen quarks and gluons, i.e., to describe the hadronic world from first principles. But this is a formidable task because of the color confinement phenomenon, the dynamical mechanism of which is not yet fully understood, and therefore the confinement problem remains unsolved up to the present days [2]. It prevents colored quarks and gluons to be experimentally detected as asymptotic states, which are colorless, by definition, so color confinement is permanent and absolute [1]. At present, there is no doubt left that color confinement as well as all other dynamical effects, such as spontaneous breakdown of chiral symmetry (SBCS), bound-state problems, etc., are inaccessible to the perturbation theory (PT) techniques, i.e., they are very essentially nonperturbative (NP) effects. This means that for their investigation the NP solutions, methods and approaches are needed to be found, used and developed. This is especially necessary taking into account that the above-mentioned NP effects are low-energy/momentum (large distances) phenomena, and, as it is well known, the PT methods, in general, fail to investigate them.

The surprising fact is that after more than thirty years of QCD, we still don't know the interaction between quarks and gluons. To know it means that one knows the full gluon and quark propagators, the quark-gluon and the pure gluon vertices. In the weak coupling limit or in the case of heavy quarks only this interaction is known. In the first case all the above-mentioned lower and higher Green's functions (propagators and vertices, respectively) become effectively free ones multiplied by the renormalization group corresponding PT logarithm improvements. In the case of heavy quarks all the Green's functions can be approximated by their free counterparts from the very beginning. In general, the Green's functions are essentially different from their free counterparts (substantially modified) due to the response of the highly nontrivial structure of the true QCD vacuum. It is just this response which is taken into account by the full ("dressed") propagators and vertices (it can be neglected in the weak coupling limit or for heavy quarks). That is the main reason why they are still unknown, and the confinement problem is not yet solved. In

*gogohia@rmki.kfki.hu

other words, it is not enough to know the Lagrangian of the theory. In QCD it is also necessary and important to know the true structure of its ground state (also there might be symmetries of the Lagrangian which do not coincide with symmetries of the vacuum). This knowledge can only come from the investigation of a general system of the dynamical equations of motion, the so-called Schwinger-Dyson (SD) system of equations [1, 3, 4], to which all the Green's functions should satisfy. To solve this system means to establish all the QCD Green's functions, and thus to establish the structure of the true QCD ground state as well. So, the SD system of equations is the only place where the color confinement problem can generally be solved. In this work, we will establish the interaction between quarks, more precisely the exact IR structure of the full gluon propagator, which is responsible for color confinement. In part II of our general investigation of the color confinement problem, we will establish the quark-gluon vertex needed for its solution. The Slavnov-Taylor (ST) identities for the pure gluon vertices will be investigated there as well (for references see below).

The full dynamical information of any quantum field gauge theory such as QCD is contained in the corresponding quantum equations of motion, the above-mentioned SD system of equations. It is a highly nonlinear, strongly coupled system of four-dimensional integral equations. To solve this system means to solve QCD itself and vice versa. The Bethe-Salpeter (BS) integral equations for the bound-state amplitudes [5] should be also included into this system. The kernels and scattering amplitudes of these integral equations are determined by an infinite series of the corresponding multi-loop skeleton diagrams. It is a general feature of nonlinear systems that the number of solutions (if any) cannot be fixed *a priori*. Although this system of dynamical equations can be reproduced by expansion around the free field vacuum, the final equations make no reference to the vacuum of PT. They are sufficiently general and should be treated beyond PT [1]. These equations should be also complemented by the corresponding ST identities [1, 3, 4, 6, 7, 8], which, in general, relate lower and higher Green's functions to each other. These identities are consequences of the exact gauge invariance and therefore "*are exact constraints on any solution to QCD*" [1]. The low-energy/momentum region interesting for confinement, SBCS, etc., is usually under the control of these identities. Precisely the SD system of dynamical equations, complemented by the ST identities, can serve as an adequate and effective tool for the NP approach to QCD [1, 9, 10, 11, 12].

However, two necessary requirements should be imposed over any solutions to QCD within its dynamical equations approach. As emphasized above, the SD system of equations is a highly nonlinear, very complicated and strongly coupled system of equations, so there is definitely "*no hope for an exact solution(s)*" [1, 9]. In this connection, let us remind that even in the case of Quantum Electrodynamics (QED) no exact solution(s) is known, though it is a much simpler gauge theory than QCD. Due to the above-mentioned complexity, the SD system of equations may contain even more additional information, which would only complicate the solution of this or that physical problem. The SD system of equations is, in fact, an infinite chain of the relations between different propagators, vertices and scattering kernels. Thus, truncations (and approximations) are inevitable in order to formulate a closed set of equations, say, in the quark sector. Different truncations could lead to qualitatively different solutions. QCD contains many sectors of different nature, such as quark, ghost, Yang-Mills (YM), Nambu-Goldstone (NG), BS, etc. However, making use of some truncation scheme in one sector, it is necessary to be sure that nothing is going wrong in other sectors.

So, the first necessary requirement is : any truncation scheme in QCD should be self – consistent. The main tool to maintain the self-consistent treatment of different sectors in QCD is, of course, the use of different ST identities, as it has been already underlined above. The importance of the self-consistent treatment of the SD system of equations within any truncation scheme, in particular ladder approximation, for the first time has been emphasized by Maskawa and Nakajima and by Adler in their pioneering papers [13, 14]. The next significant step in this direction has been done in Ref. [15], while in our papers [16] it has been finally proven that the ladder approximation in the whole energy-momentum range is not self-consistent.

The second necessary requirement is a manifest gauge invariance, i.e., to use and develop only those self-consistent truncation schemes (approaches), which do not depend explicitly on the gauge fixing parameter. This is crucially important in QCD just because there is no hope for exact solution(s). In this connection, a few remarks are in order. In QED the explicit gauge dependence is not a problem. In order to calculate physical observables in this theory, one needs to multiply the corresponding elements of S -matrix by the conserved currents, which immediately eliminates the dependence on the gauge fixing parameter (i.e., the longitudinal component of the photon propagator does not contribute to S -matrix). Obviously, this is impossible in QCD due to its non-Abelian character. First of all, the transverse and longitudinal components of the gluon propagator do interact with each other. Secondly, the current to which the gauge field is coupled is not conserved [17, 18]. In QCD to find a calculation scheme (approach), which does not depend explicitly on the gauge fixing parameter from the very beginning, is crucially important, indeed.

It is worth explaining this point in more detail. Let us consider for this purpose, for example the π^+ meson propagator, which can be written down analytically as follows:

$$D_{\pi^+}(p) \sim \int d^4q \text{Tr} \left[\gamma_5 S_u(p+q) \gamma_5 S_{\bar{d}}(q) \right], \quad (1.1)$$

where all numerical factors are suppressed as unimportant for our discussion. For simplicity, the BS bound-state amplitudes for the π^+ meson are replaced by the γ_5 matrices. Let us emphasize now that the left-hand-side of this relation is, by definition, a manifestly gauge-invariant quantity, since it describes the propagation of a physical particle, the π^+ meson. However, single quark propagators, which appear in the right-hand-side of this relation, are gauge-dependent quantities, i.e., they formally depend explicitly on the gauge choice. First of all, this dependence comes from the gluon propagator, which appears in the quark self-energy. That is why a gauge-invariant approach is necessary in order to calculate quark propagators, i.e., only those quark propagators, calculated within a manifestly gauge-invariant approach, are to be used to form gauge-invariant composite propagators such as the meson propagator [19]. This guarantees that the right-hand-side of the relation (1.1) will be manifestly gauge-invariant, as it is required by its left-hand-side. Thus, all the solutions to the SD dynamical equations of motion, based on the particular gauge choice (the so-called gauge artifact solutions) should be ruled out. In other words, such kind of the solutions for the quark propagator are not legitimated to use in order to form gauge-invariant meson propagator, or to calculate any other physical observables in QCD, in particular, in low-energy QCD. Let us stress once more that by "gauge artifact solutions" we mean solutions, which are due to the particular gauge choice, i.e., they cannot be obtained by choosing some another numerical value of the gauge fixing parameter (for more detail remarks see Subsec. D in Sec. 2).

Our approach to low-energy QCD is formulated in the framework of the SD system of dynamical equations mentioned above. We propose and develop an approach which, on one hand, is self-consistent, and, on the other hand, is manifestly gauge-invariant. It allows one to calculate hadron properties from first principles, i.e., from the underlying dynamical theory of interactions between quark and gluons only. It is based on the existence and the importance of quantum fluctuations and excitations of the infrared (IR) degrees of freedom in the QCD ground state. Our approach has been recently applied to two-dimensional (2D) covariant gauge QCD, which has been proven to confine quarks [20, 21]. Its axial gauge counterpart has been already investigated in 't Hooft's pioneering paper [22] within precisely quark SD and BS equations approach. 2D covariant gauge QCD turned out to be an appropriate theory in order to be generalized to four-dimensional (4D) QCD. However, there exist, at least, four important distinctions between 2D and 4D QCD in any gauge. First of all, 2D QCD has explicitly (in the Lagrangian) a fundamental mass scale parameter, which is nothing else but the coupling constant. In 4D QCD the coupling is dimensionless and Lagrangian of this theory does not explicitly contain a fundamental scale (the mass gap). Second, the IR singularity of the free gluon propagator in 2D QCD is severe one (in any gauge, of course), i.e., it is NP from the very beginning [20, 21]. In 4D QCD the IR singularity of the free gluon propagator is not severe, i.e., it is PT one (for the exact definitions of both the PT and the NP IR singularities see below). So, there is no justification to use it for the description of the IR region in 4D QCD ground state. The third important distinction is that 4D QCD has a nontrivial PT phase, while in 2D QCD it is simple. And finally the 4D QCD ground state is much more rich and complicated than its almost trivial 2D counterpart. Let us also point out an interesting discussion of the Coleman theorem concerning the absence of the NG bosons in 2D quantum field theories (the sine-Gordon and Thirring models) in Ref. [23].

In our general approach to 4D QCD, i.e., to QCD itself, we have elaborated on all the above-mentioned distinctions from 2D QCD. It will consist of a few parts. In part I, we will formulate our approach to the YM sector in QCD in all details. We establish uniquely and exactly the IR structure of the full gluon propagator, fixing thus the first part of the interaction between quarks. This automatically requires the existence of a mass gap, which is responsible for the NP dynamics in the true QCD ground state. Moreover, we formulate the gluon confinement criterion in a manifestly gauge-invariant way. In part II, we establish the second part of the above-mentioned interaction, namely the needed quark-gluon proper vertex. This will be done with the help of the corresponding ST identity. A closed set of equations for the quark propagator will be derived. By the consideration of the whole system of the SD equations and corresponding ST identities, the IR renormalizability of QCD will be proven under some conditions. In part III, a closed set of equations for the quark propagator, derived in part II, will be solved. We will show explicitly that our approach to low-energy QCD implies quark confinement and SBCS without involving any extra degrees of freedom. We need only the underlying dynamics of QCD – the self-interaction of massless gluons. In part IV, the NG sector of QCD will be numerically evaluated. Developing the so-called chiral PT at the fundamental quark level, we will be able to fix the BS bound-state amplitude in the chiral limit for the NG particles. It will also allow one to derive a new formula for the pion decay constant. The part V will be devoted to the investigation of the QCD ground state itself. We will show how to calculate correctly the truly NP vacuum energy density. It is the most important characteristics of the QCD ground state. The BS sector (i.e., the bound-state problem) will be investigated in the next part VI. We will show that the bound-state problem will be always reduced to an algebraic problem, i.e., it becomes tractable within our approach.

This paper (hereafter the above-mentioned part I, which plays a central role in our general approach to low-energy QCD) is organized as follows. In Sec. 2, we investigate the general properties of the SD equation for the full gluon propagator in QCD. It is also explicitly shown there how severe IR singularities and a mass gap may appear in the true QCD vacuum due to the self-interaction of massless gluons. In Sec. 3, we introduce a few useful formulae from the distribution theory (DT) [24], complemented by the dimensional regularization (DR) method [25]. They show

how severe IR singularities should be correctly put under firm mathematical control. Sec. 4 is devoted to establishing the general structure of the full gluon propagator in the IR region. We will show that it is given by the special Laurent expansion as an infinite sum over all severe power-type IR singularities possible in nD QCD. An important case of QCD itself is considered in Sec. 5. The general IR renormalization program is formulated in Sec. 6. In this section, we perform the IR renormalization of the gluon SD equation. It has been proven that by the IR renormalization of a mass gap only, all severe IR singularities can be removed from the full gluon propagator. In Sec. 7, we formulate the zero momentum modes enhancement (ZMME) model of the true QCD ground state. Within it, we thus determine a pure gluon part of the interaction between quarks. A gluon confinement criterion is formulated in a manifestly gauge-invariant way. In the same way, we also define the intrinsically NP (INP) phase in QCD. In Sec. 8, the solution of the gluon SD equation within our approach is given. In Sec. 9, the general discussion is present, while in Sec. 10 we summarize our conclusions. We investigate the gluon SD equation for the PT part of the full gluon propagator in the appendix A.

II. GLUON PROPAGATOR

In order to investigate the problem of the true QCD ground state structure, let us start first with one of the main objects in the YM sector. The two-point Green's function, describing the full gluon propagator, is (Euclidean signature here and everywhere below)

$$D_{\mu\nu}(q) = i \{T_{\mu\nu}(q)d(q^2, \xi) + \xi L_{\mu\nu}(q)\} \frac{1}{q^2}, \quad (2.1)$$

where ξ is the gauge fixing parameter ($\xi = 0$ - Landau gauge and $\xi = 1$ - Feynman gauge) and

$$T_{\mu\nu}(q) = \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} = \delta_{\mu\nu} - L_{\mu\nu}(q). \quad (2.2)$$

Evidently, $T_{\mu\nu}(q)$ is the transverse (physical) component of the full gluon propagator, while $L_{\mu\nu}(q)$ is its longitudinal (unphysical) one. The free gluon propagator is obtained by setting simply the full gluon form factor $d(q^2, \xi) = 1$ in Eq. (2.1), i.e.,

$$D_{\mu\nu}^0(q) = i \{T_{\mu\nu}(q) + \xi L_{\mu\nu}(q)\} \frac{1}{q^2}. \quad (2.3)$$

The solutions of the SD equation for the full gluon propagator (2.1) are supposed to reflect the complexity of the quantum structure of the QCD ground state. Just this determines one of the central roles of the full gluon propagator in the SD system of equations [26]. The SD equation for the full gluon propagator (see Eq. (2.4)) is a highly nonlinear system of four-dimensional integrals, containing many different, unknown in general, propagators and vertices, which, in their turn, satisfy too complicated integral equations, containing different scattering amplitudes and kernels, so there is no hope for exact solution(s). However, in any case the solutions of this equation can be distinguished from each other by their behavior in the IR limit, describing thus many (several) different types of quantum excitations and fluctuations of gluon field configurations in the QCD vacuum. Evidently, not all of them can reflect the real structure of the QCD vacuum, for example the gauge artifact solutions (see Subsec. E). The ultraviolet (UV) limit of these solutions is uniquely determined by asymptotic freedom (AF) [27].

The deep IR asymptotics of the full gluon propagator can be generally classified into the two different types: singular, which means that the above-mentioned ZMME effect takes place in the NP QCD vacuum, or smooth, which means that the full gluon propagator is IR finite or even is IR vanishing. Formally, the full gluon propagator (2.1) has an exact power-type IR singularity, $1/q^2$, which is due to its longitudinal component. This is the IR singularity of the free gluon propagator, see Eq. (2.3). By the ZMME effect we mean, in general, the IR singularities, which are more severe than $1/q^2$ (see also Subsec. C). Evidently, the singular asymptotics is possible at any value of the gauge fixing parameter. At the same time, the smooth behavior of the full gluon propagator (2.1) in the IR becomes formally possible either by choosing the Landau gauge $\xi = 0$ from the very beginning, or by removing the longitudinal (unphysical) component of the full gluon propagator with the help of ghost degrees of freedom [1, 17, 18] (for more detail discussion see Subsec. D).

However, any deviation in the behavior of the full gluon propagator in the IR domain from the free one automatically assumes its dependence on a scale parameter (at least one) different, in general, from the QCD asymptotic scale

parameter Λ_{QCD} . It can be considered as responsible for the NP dynamics (in the IR region) in the QCD vacuum. If QCD itself is a confining theory, then such a characteristic scale is very likely to exist. This is very similar to AF, which requires the above-mentioned asymptotic scale parameter Λ_{QCD} associated with nontrivial PT dynamics in the UV region (AF, scale violation, determining thus the deviation in the behavior of the full gluon propagator from the free one in the UV domain). In this connection it is worth emphasizing that, being numerically a few hundred MeV only, it cannot survive in the UV limit. This means that none of the finite scale parameters, in particular Λ_{QCD} , can be determined by PT QCD. It should come from the IR region, so it is NP by origin. How to establish a possible relation between these two independent scale parameters was shown in our paper [28]. Despite the fact that the PT vacuum cannot be the true QCD ground state [29], nevertheless, the existence of such kind of a relation is a manifestation that "the problems encountered in perturbation theory are not mere mathematical artifacts but rather signify deep properties of the full theory" [30].

The message that we have tried to convey is that precisely AF clearly indicates the existence of the NP phase with its own characteristic scale parameter in the full QCD.

A. Gluon SD equation

The general structure of the SD equation for the full gluon propagator can be written down symbolically as follows (for our purposes it is more convenient to consider the SD equation for the full gluon propagator and not for its inverse, as usual):

$$D(q) = D^0(q) - D^0(q)T_{gh}(q)D(q) - D^0(q)T_q(q)D(q) + D^0(q)T_g[D](q)D(q). \quad (2.4)$$

Here and in some places below, we omit the dependence on the Dirac indices, for simplicity. $T_{gh}(q)$ and $T_q(q)$ describe the ghost and quark skeleton loop contributions into the gluon propagator. They do not contain the full gluon propagators by themselves. A pure gluon contribution $T_g[D](q)$ is a sum of four pure gluon skeleton loops, and consequently they explicitly contain the full gluon propagators. Precisely this makes the gluon SD equation highly nonlinear (NL), and this is one of the reasons why it cannot be solved exactly. However, its linear part, which contains only ghost and quark skeleton loops, can be summed up, so Eq. (2.4) becomes

$$D(q) = \tilde{D}^0(q) + \tilde{D}^0(q)T_g[D](q)D(q) = \tilde{D}^0(q) + D^{NL}(q), \quad (2.5)$$

with $\tilde{D}^0(q)$ being a modified free gluon propagator as follows:

$$\tilde{D}^0(q) = \frac{D^0(q)}{1 + [T_{gh}(q) + T_q(q)]D^0(q)}, \quad (2.6)$$

where

$$T_{gh}(q) = g^2 \int \frac{id^4k}{(2\pi)^4} k_\nu G(k) G(k-q) G_\mu(k-q, q), \quad (2.7)$$

$$T_q(q) = -g^2 \int \frac{id^4p}{(2\pi)^4} Tr[\gamma_\nu S(p-q) \Gamma_\mu(p-q, q) S(p)]. \quad (2.8)$$

Let us note in advance that, in general, these quantities can be decomposed as follows:

$$T_{gh}(q) \equiv T_{\mu\nu}^{gh}(q) = \delta_{\mu\nu} q^2 T_{gh}^{(1)}(q^2) + q_\mu q_\nu T_{gh}^{(2)}(q^2), \quad (2.9)$$

$$T_q(q) \equiv T_{\mu\nu}^q(q) = \delta_{\mu\nu} q^2 T_q^{(1)}(q^2) + q_\mu q_\nu T_q^{(2)}(q^2), \quad (2.10)$$

where all invariant functions $T_{gh}^{(n)}(q^2)$ and $T_q^{(n)}(q^2)$ at $n = 1, 2$ are dimensionless with a regular behavior at zero (they include the dependence on the coupling constant squared g^2). In this connection a few remarks are in order. Due

to the definition $q_\mu q_\nu = q^2 L_{\mu\nu}$ (see relation (2.2)), instead of the independent structures $\delta_{\mu\nu}$ and $q_\mu q_\nu$ in Eqs. (2.9) and (2.10), one can use $T_{\mu\nu}$ and $L_{\mu\nu}$ as independent structures with their own invariant functions. For simplicity, we assume here and everywhere below that all integrals are finite, and consequently all invariant functions are also finite at zero. Anyway, how to render them finite is well known procedure (see, for example Refs. [1, 17, 18, 31]).

From a technical point of view it is convenient to use the free gluon propagator (2.3) in the Feynman gauge ($\xi = 1$), i.e., $D_{\mu\nu}^0(q) = \delta_{\mu\nu}(i/q^2)$. Then from Eq. (2.6) it follows

$$\tilde{D}^0(q) = D^0(q)A(q^2), \quad (2.11)$$

where

$$A(q^2) = \frac{1}{1 + T(q^2)}, \quad (2.12)$$

and $T(q^2)$ is regular at zero. Obviously, it is a combination of the previous ghost $T_{gh}^{(n)}(q^2)$ and quark $T_q^{(n)}(q^2)$ at $n = 1, 2$ invariant dimensionless functions (it includes the dependence on the coupling constant squared again and the gauge fixing parameter as well in the general case (i.e., when $D^0(q)$ is given by Eq. (2.3)). Since $A(q^2)$ is finite at zero, the IR singularity of the linear part of the full gluon propagator is completely determined by the power-type IR singularity of the free gluon propagator, as it follows from Eq. (2.11), i.e., $\tilde{D}^0(q) = A(0)D^0(q)$, $q^2 \rightarrow 0$. We are especially interested in the structure of the full gluon propagator in the IR region, so the relation (2.11) will be used as an input in the direct iteration solution of the gluon SD equation (2.5). Evidently, this form of the gluon SD equation makes it possible to take into account automatically ghost and quark degrees of freedom in all orders of linear PT. On the other hand, it emphasizes the important role of the pure gluon contribution (i.e., YM one), which forms its NL part.

Let us present now the NL pure gluon part, which was symbolically denoted as $T_g[D](q)$ in the gluon SD Eq. (2.5). As mentioned above, it is a sum of four terms, namely

$$T_g[D](q) = \frac{1}{2}T_t + \frac{1}{2}T_1(q) + \frac{1}{2}T_2(q) + \frac{1}{6}T'_2(q), \quad (2.13)$$

where the corresponding quantities are given explicitly below

$$T_t = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 D(q_1), \quad (2.14)$$

$$T_1(q) = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_3^0(q, -q_1, q_1 - q) T_3(-q, q_1, q - q_1) D(q_1) D(q - q_1), \quad (2.15)$$

$$T_2(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q_3, q_2 - q_3) T_3(-q, q_1, q_3 - q_2) D(q_1) D(-q_2) D(q_3) D(q_3 - q_2), \quad (2.16)$$

$$T'_2(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q_3) D(q_1) D(-q_2) D(q_3). \quad (2.17)$$

In the last two equations $q - q_1 + q_2 - q_3 = 0$ is assumed as usual. The T_t term, which is given in Eq. (2.14), is the so-called tadpole term constant contribution into the gluon propagator (gluon self-energy). The $T_1(q)$ term describes the one-loop skeleton contribution, depending on the three-gluon vertices only. The $T_2(q)$ term describes the two-loop skeleton contribution, depending on the three- and four-gluon vertices, while the $T'_2(q)$ term describes the two-loop skeleton contribution, depending on the four-gluon vertices only.

The formal iteration solution of Eq. (2.5) looks like

$$D(q) = D^{(0)}(q) + \sum_{k=1}^{\infty} D^{(k)}(q) = D^{(0)}(q) + \sum_{k=1}^{\infty} \left[D^{(0)}(q) T_g \left[\sum_{m=0}^{k-1} D^{(m)} \right] (q) \left(\sum_{m=0}^{k-1} D^{(m)}(q) \right) - \sum_{m=1}^{k-1} D^{(m)}(q) \right], \quad (2.18)$$

where, for example explicitly the first four terms are:

$$\begin{aligned}
D^{(0)}(q) &= \tilde{D}^0(q), \\
D^{(1)}(q) &= \tilde{D}^0(q)T_g[\tilde{D}^0](q)\tilde{D}^0(q), \\
D^{(2)}(q) &= \tilde{D}^0(q)T_g[\tilde{D}^0 + D^{(1)}](q)(\tilde{D}^0(q) + D^{(1)}(q)) - D^{(1)}(q), \\
D^{(3)}(q) &= \tilde{D}^0(q)T_g[\tilde{D}^0 + D^{(1)} + D^{(2)}](q)(\tilde{D}^0(q) + D^{(1)}(q) + D^{(2)}(q)) - D^{(1)}(q) - D^{(2)}(q),
\end{aligned} \tag{2.19}$$

and so on. It is worth mentioning that the order of iteration does not coincide with the order of PT in the coupling constant squared. For example, any iteration (even zero) in Eq. (2.18) contains ghost and quark degrees of freedom in all orders of PT, as underlined above. In other words, the iteration solution (2.18) is a general one, since the skeleton loop contributions (skeleton diagrams) are to be iterated (the so-called general iteration solution). In principle, it should be distinguished from the pure PT iteration solution, i.e., from the expansion in powers of the coupling constant squared. In this case the pure PT diagrams (with free propagators and point-like vertices) are to be iterated. Of course, there is no hope to find solution in a closed form, i.e., to sum up an infinite series presented in Eq. (2.18). However, for future purpose it is useful to note its most important algebraic feature. As it follows from Eqs. (2.19) each subsequent iteration contains all the preceding ones. Below we will establish the dynamical context of the general iteration solution (2.18).

B. The deep IR structure of the gluon propagator

In order to investigate the IR structure of the full gluon propagator it is instructive to start from the investigation of the linear part of the gluon SD equation in the straightforward $q = 0$ limit. Since the full gluon propagator has, in general, the IR singularity $1/q^2$ (see Eq. (2.1)), the replacement $1/q^2 \rightarrow 1/(q^2 + i\epsilon)$ is understood in order to assign a mathematical meaning to this limit (this ϵ is not to be mixed up with the IR regularization parameter introduced in Sec. 3). The corresponding skeleton loop integral, which contains quark degrees of freedom (2.8) in this limit becomes

$$T_q(0) = -g^2 \int \frac{id^4 p}{(2\pi)^4} Tr[\gamma_\nu S(p)\Gamma_\mu(p, 0)S(p)]. \tag{2.20}$$

It is easy to see that this integral does not exhibit any singularities in the integrand at very small values of the skeleton loop variable. This number may be infinite in the UV limit, and its removal is the subject to the corresponding UV renormalization procedure, as mentioned above.

In the same way the ghost skeleton loop integral (2.7) becomes

$$T_{gh}(0) = g^2 \int \frac{id^4 k}{(2\pi)^4} k_\nu G(k)G(k)G_\mu(k, 0). \tag{2.21}$$

At first sight an additional singularity at very small values of the skeleton loop variable will appear because of the second ghost propagator. However, this is not the case. The ghost-gluon vertex $G_\mu(k, 0)$ is the linear function of its argument and the combination $k_\nu k_\mu$ will cancel this additional singularity. So, it is finite in the IR region and its removal is again the subject to the UV renormalization program.

Since the NL part of the gluon SD equation starts from the tadpole term (2.14), which does depend on the external gluon momentum q at all, let us consider the skeleton loop integral (2.15) first. In the exact $q = 0$ limit it is

$$T_1(0) = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_3^0(0, -q_1, q_1)T_3(0, q_1, -q_1)D(q_1)D(-q_1). \tag{2.22}$$

An additional singularity due to $D(-q_1) = D(q_1)$ will appear. However, the three-gluon vertices which are present in the nominator being the linear functions of their arguments will cancel this additional IR singularity. Thus, this skeleton loop integral is finite in the deep IR region.

The two-loop skeleton integral (2.16) in the exact $q = 0$ limit becomes

$$T_2(0) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, -q_1 + q_2, q_1)T_3(0, q_1, -q_1)D(q_1)D(-q_2)D(-q_1 + q_2)D(-q_1). \tag{2.23}$$

Again an additional singularities in the integration over the very small values of the loop variables q_1 and q_2 due to the full gluon propagators $D(-q_1 + q_2)$ and $D(-q_1)$ are to appear. However, contrary to the previous case these singularities might not be cancelled by terms which come from the corresponding three-gluon vertices at the very small values of all the gluon momenta involved (the number of the three-gluon vertices might not be enough). So, this skeleton loop integral can be source of the additional IR singularities with respect to the small values of the external gluon momentum q (however, see remarks below in Subsec. E).

The most strict evidence of unavoidable occurrence of the additional IR singularities in the full gluon propagator is given by the last two-loop skeleton integral (2.17). In the exact $q = 0$ limit it is

$$T_2'(0) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(0, q_1, -q_2, -q_1 + q_2) D(q_1) D(-q_2) D(-q_1 + q_2). \quad (2.24)$$

Again an additional singularities will appear due to $D(-q_1 + q_2)$ in the integration over the very small values of the loop variables q_1 and q_2 . **The important observation, however, is that they cannot be cancelled by the corresponding terms from the nominator, for sure, since the full four-gluon vertex, when all the gluon momenta involved go to zero, will be reduced to the corresponding point-like four-gluon vertex, which does not depend on the gluon momenta involved at all.** Thus, the straightforward $q = 0$ limit is certainly dangerous in this case, and more sophisticated method is needed to investigate the region of all the small gluon momenta involved (see below). The IR singular structure of this two-loop skeleton integral is the principal distinction from all other skeleton loop integrals considered above. Let us also make once thing perfectly clear. Due to the above-mentioned singular structure this integral implicitly contains the corresponding mass scale parameter (the mass gap). In other words, the mass gap is hidden in the initial skeleton integral (2.17) at nonzero q . They (the mass gap and an additional singularities) will show up when q goes to zero. How to extract them explicitly from the corresponding Feynman diagram see next Subsec.

C. The explicit functional estimate

Let us now establish a type of a possible functional dependence of the full gluon propagator in the IR region. For this purpose it is convenient to start with the gluon SD equation (2.5). Up to the first iteration it becomes

$$D(q) = \tilde{D}^0(q) + \tilde{D}^0(q) T_g[D](q) D(q) = \tilde{D}^0(q) + \tilde{D}^0(q) T_g[\tilde{D}^0](q) \tilde{D}^0(q) + \dots, \quad (2.25)$$

where we will use Eq. (2.11) for the modified free gluon propagator in the Feynman gauge in what follows. In order to clearly separate an additional IR singularities in the skeleton loop integral (2.17) it is sufficient to explicitly consider it at the order g^4 , for which we should put $T_4 = T_4^0$. To this order the two-loop contribution (2.17) then becomes

$$T_2'(q) \equiv T_{\nu_1 \mu_1}^{(2)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_{\nu_1 \rho \lambda \sigma}^0 T_{\mu_1 \rho_1 \lambda_1 \sigma_1}^0 \tilde{D}_{\rho \rho_1}^0(q_1) \tilde{D}_{\lambda \lambda_1}^0(-q_2) \tilde{D}_{\sigma \sigma_1}^0(q - q_1 + q_2), \quad (2.26)$$

where it is assumed that the summation over color group factors has been already done and is included into the coupling constant (as well as some other finite numerical factors, which can appear as results of the integration, see below), since these numbers are not important. The summation over Dirac indices then yields

$$T_{\nu_1 \mu_1}^{(2)}(q) = -i \delta_{\nu_1 \mu_1} g^4 \int id^4 q_1 \int id^4 q_2 \frac{A(q_1^2) A(q_2^2) A((q - q_1 + q_2)^2)}{q_1^2 q_2^2 (q - q_1 + q_2)^2} = -i \delta_{\nu_1 \mu_1} g^4 F_2'(q^2), \quad (2.27)$$

where we introduce

$$F_2'(q^2) = \int id^4 q_1 \int id^4 q_2 \frac{A(q_1^2) A(q_2^2) A((q - q_1 + q_2)^2)}{q_1^2 q_2^2 (q - q_1 + q_2)^2}. \quad (2.28)$$

As emphasized above, this integral possesses very distinctive and important feature, namely it exhibits an additional singularities in the integration over the very small values of the loop variables q_1 and q_2 when the straightforward $q = 0$ limit for the external gluon momentum is undertaken, so it should not be finite in this limit. In this connection,

let us remind that the A -function is regular at zero. Thus, the exact $q = 0$ limit is dangerous, since this is the IR singularity of the full gluon propagator as well (see Eq. (2.1)). Let us also emphasize once more that the existence of the additional IR singularities assumes the existence of the corresponding mass scale parameter, at least one, (the mass gap), which is "hidden" in this integral. As mentioned above, more sophisticated method is needed to detect the above-mentioned additional IR singularities, and hence the existence of the corresponding mass gap in the deep IR structure of the full gluon propagator.

For this purpose and in order to introduce a mass gap, which determines the deviation of the full gluon propagator from the free one in the IR region (due to the above-mentioned additional IR singularities) at the level of the separate diagram (contribution), let us present the last integral as a sum of four terms, namely

$$F'_2(q^2) = \sum_{n=1}^{n=4} F_2^{(n)}(q^2), \quad (2.29)$$

where

$$F_2^{(1)}(q^2) = \int_0^{\Delta^2} id^4 q_1 \int_0^{\Delta^2} id^4 q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2 (q - q_1 + q_2)^2}, \quad (2.30)$$

$$F_2^{(2)}(q^2) = \int_{\Delta^2}^{\infty} id^4 q_1 \int_0^{\Delta^2} id^4 q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2 (q - q_1 + q_2)^2}, \quad (2.31)$$

$$F_3^{(3)}(q^2) = \int_0^{\Delta^2} id^4 q_1 \int_{\Delta^2}^{\infty} id^4 q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2 (q - q_1 + q_2)^2}, \quad (2.32)$$

$$F_2^{(4)}(q^2) = \int_{\Delta^2}^{\infty} id^4 q_1 \int_{\Delta^2}^{\infty} id^4 q_2 \frac{A(q_1^2)A(q_2^2)A((q - q_1 + q_2)^2)}{q_1^2 q_2^2 (q - q_1 + q_2)^2}, \quad (2.33)$$

where not loosing generality we introduced the common mass gap squared Δ^2 for both loop variables q_1^2 and q_2^2 . The integration over angular variables is assumed.

As mentioned above, we are especially interested in the region of all the small gluon momenta involved, i.e., $q \approx q_1 \approx q_2 \approx 0$. However, in Eq. (2.30) we can formally consider the variables q_1 and q_2 as much smaller than the small gluon momentum q , i.e., to approximate $q_1 \approx \delta_1 q$, $q_2 \approx \delta_2 q$, so that $q - q_1 + q_2 \approx q(1 + \delta)$, where $\delta = \delta_2 - \delta_1$. To leading order in δ , one obtains

$$F_2^{(1)}(q^2) = -\frac{A(q^2)}{q^2} \int_0^{\Delta^2} dq_1^2 \int_0^{\Delta^2} dq_2^2 A(q_1^2)A(q_2^2), \quad (2.34)$$

where all the finite numbers after the trivial integration over angular variables will be included into the numerical factors below, for simplicity. Since q^2 is small, we can replace the dimensionless function $A(q^2)$ by its Taylor expansion as follows: $A(q^2) = A(0) + a_1(q^2/\Delta^2) + O(q^4)$. Introducing further dimensionless variables $q_1^2 = x_1 \Delta^2$ and $q_2^2 = x_2 \Delta^2$, one finally obtains

$$F_2^{(1)}(q^2) = -\frac{\Delta^4}{q^2} c_1 - \Delta^2 c'_1 + O(q^2), \quad (2.35)$$

where

$$c_1 = A(0) \int_0^1 dx_1 A(x_1) \int_0^1 dx_2 A(x_2), \quad (2.36)$$

and $c'_1 = a_1(c_1/A(0))$. The both numbers are obviously finite.

In Eq. (2.31) it makes sense to approximate $q_2 \approx \delta_3 q_1$, $q \approx \delta_4 q_1$, so that $q - q_1 + q_2 \approx q_1(1 + \tilde{\delta})$, where $\tilde{\delta} = \delta_4 - \delta_3$. To leading order in $\tilde{\delta}$ and omitting some algebra, one finally obtains

$$F_2'^{(2)}(q^2) = -\Delta^2 c_2(\nu) + O(q^2), \quad (2.37)$$

where

$$c_2(\nu) = \int_1^\nu \frac{dx_1}{x_1} A^2(x_1) \int_0^1 dx_2 A(x_2), \quad (2.38)$$

and ν is the dimensionless auxiliary UV cut-off.

In Eq. (2.32) it makes sense to approximate $q_1 \approx \delta_5 q_2$, $q \approx \delta_6 q_2$, so that $q - q_1 + q_2 \approx q_2(1 + \bar{\delta})$, where $\bar{\delta} = \delta_5 + \delta_6$. To leading order in $\bar{\delta}$ and similar to the previous case, one obtains

$$F_2'^{(3)}(q^2) = -\Delta^2 c_3(\nu) + O(q^2), \quad (2.39)$$

where

$$c_3(\nu) = \int_1^\nu \frac{dx_2}{x_2} A^2(x_2) \int_0^1 dx_1 A(x_1). \quad (2.40)$$

The last term (2.33) is left unchanged, since all loop variables are big. Conventionally, we will call it as the PT part of the contribution (diagram), i.e., denoting $F_2'^{(4)}(q^2)$ as $F_2'^{PT}(q^2)$. Since $A(x)$ is regular at zero, the both integrals in Eqs. (2.38) and (2.40) are logarithmically divergent. Also if one wants to neglect ghost and quark degrees of freedom one only needs to replace everywhere the A -function by unity (see Eq. (2.12)).

Summing up all terms, one obtains

$$T_2'(q) \equiv T_{\nu_1 \mu_1}'^{(2)}(q) = i\delta_{\nu_1 \mu_1} \left[\frac{\Delta^4}{q^2} c_1 + \Delta^2 (c_2(\nu) + c_3(\nu)) \right] g^4 + O(q^2). \quad (2.41)$$

The term $F_2'^{PT}(q^2)$ is hidden in terms $O(q^2)$. Here the characteristic mass scale parameter Δ^2 is responsible for the nontrivial dynamics in the IR domain. Let us also emphasize that the limit $\nu \rightarrow \infty$ should be taken at the final stage. Anyway, the finite constant c_1' from the expansion (2.35) has been already suppressed in comparison with the logarithmically divergent integrals $c_2(\nu)$ and $c_3(\nu)$ even at this stage. So, the integral (2.28) and hence the original integral (2.26) is divergent in the exact $q = 0$ limit, indeed. In other words, these singularities with respect to the external gluon momentum q will show explicitly up if and only if it goes to zero. Evidently, the integral (2.30) is an example of overlapping IR divergences, nevertheless, it plays no important role in the deep IR structure of the gluon propagator. First of all, the finite constant contribution is to be neglected as mentioned above. From the IR renormalization procedure it will follow (see Secs. VI and VII) that the terms of the order $O(\Delta^4)$ will be suppressed.

The constant tadpole term (2.14) produces the contribution as follows: $T_t \equiv T_{\nu_1 \mu_1}^t = -i\delta_{\nu_1 \mu_1} \Delta^2 c_t(\nu) g^2$, where $c_t(\nu) = \int_0^\nu dx_1 A(x_1)$. Formally it can be identically decomposed into the two parts by introducing $c_t(\nu) = c_t(\nu) + c_t - c_t = c_t + c_t'(\nu)$. The finite part depending on c_t can be included into the INP part of the full gluon propagator (see below), while leaving other infinite part for its PT part. Let us note, however, that in dimensional regularization this term in the pure PT iteration solution (which means $D = D^0 + \dots$) of the gluon SD equation can be generally discarded [17]. Thus, this term itself is not important at all.

Evidently, such kind of the auxiliary (but important) procedure, described in this Subsec., is appropriate only for the establishing the most singular (leading and next-to-leading) terms in the deep IR structure of the full gluon propagator. On the other hand, it makes the explicit dependence of this structure on the mass gap perfectly clear.

D. Severe IR structure of the QCD vacuum

At the NL two-loop level, i.e., at the order g^4 , there is a number of the additional diagrams, which, however, contain the three-gluon vertices (plus the two-tadpole diagram) along with the four-gluon ones. Their possible contributions into the deep IR structure of the gluon propagator are given by the estimates similar to the estimate (2.41) with

different coefficients, of course. Omitting some really tedious algebra, the full gluon propagator (2.25) up to the first iteration can be written as

$$D_{\mu\nu}(q) = i\delta_{\mu\nu} \left[\frac{\Delta^2}{(q^2)^2} a_1 + \frac{\Delta^4}{(q^2)^3} a_2 + \dots \right] + D_{\mu\nu}^{PT}(q) = D_{\mu\nu}^{INP}(q) + D_{\mu\nu}^{PT}(q), \quad (2.42)$$

where a_1 , a_2 are, in general, the short-hand notations for a sums of the different coefficients, which include the coupling constant squared in the corresponding powers. Moreover, some of these coefficients contain the divergent integrals (see, for example Eqs. (2.38) and (2.40)). Here $D_{\mu\nu}^{PT}(q)$ denotes the contribution from the PT part of the full gluon propagator, since it is of the order $O(q^{-2})$ as $q^2 \rightarrow 0$. The superscript "INP" stands for the intrinsically NP part of the full gluon propagator (for the exact definition see Secs. 4 and 7 below). Due to the distinction between the behavior of the tree- and four-gluon vertices in the deep IR domain (i.e, when all the gluon momenta involved go to zero, see discussion in Subsec. E below), the coefficients a_1 , a_2 are, in general, not zero. In other words, there is no way to cancel $D_{\mu\nu}^{INP}(q)$ by performing the functional estimate at every order of the QCD coupling constant squared. In the deep IR region the quark and ghost degrees of freedom are taken into account in all orders of linear PT numerically, i.e., they are simply numbers. As functions they can contribute into the PT part only of the full gluon propagator. So, in the first approximation the gluon propagates like Eq. (2.42) and not like the modified free one (2.11), though we just started from it.

The true QCD vacuum is really beset with severe (i.e., more singular than $1/q^2$ as $q^2 \rightarrow 0$) IR singularities if standard PT is applied. Moreover, each severe IR singularity is to be accompanied by the corresponding powers of the mass gap, responsible for the NP dynamics in the IR region. In more complicated cases of the multi-loop diagrams (i.e., the next iterations in Eq. (2.25)) more severe IR divergences will appear. The coefficients at each severe IR singularity become by themselves an infinite series in the coupling constant squared, and the coefficients of these expansions may depend on the gauge fixing parameter as well [17]. These coefficients include numerically the information about quark and ghost degrees of freedom in all orders of linear PT, as underlined above.

It is worth emphasizing, however, that the ZMME effect in the QCD vacuum, which is explicitly shown in Eq. (2.42) in the Feynman gauge, can be demonstrated in any covariant gauge, for example in the Landau one $\xi = 0$. In other words, this effect itself is gauge-invariant, though the finite sum of all the relevant diagrams in the deep IR region at the same order of the coupling constant squared may be not. Let us also remind that this effect is not something new. It has been well known for a long time from the very beginning of QCD, and it was the basis for the proposed then IR slavery (IRS) mechanism of quark confinement [1, 32, 33, 34, 35, 36]. Just this IR violent behavior makes QCD as a whole an IR unstable theory, and therefore it has no IR stable fixed point, indeed [1, 32]. **Moreover, we unambiguously identify the main source of the IR instability of QCD, namely the four-gluon vertex at the Lagrangian level and the two-loop skeleton term, which contains only the four-gluon vertices, at the level of the gluon SD equation. Precisely this interaction exhibits an additional IR singularities in the corresponding loop integrals when all the gluon momenta involved go to zero.**

The existence of a severe IR singularities automatically requires an introduction of a mass gap, responsible for the nontrivial dynamics in the IR region. This is important, since there is none explicitly present in the QCD Lagrangian (the current quark mass cannot be considered as a mass gap, since it is not renormalization group invariant). It precisely determines to what extent the full gluon propagator effectively changes its behavior from the behavior of the free one in the IR domain. The phenomenon of "dimensional transmutation" [1, 37] only supports our general conclusion that QCD may exhibit a mass, determining the characteristic scale of the NP dynamics in its ground state. Of course, such gluon field configurations, which are to be described by severely IR structure of the full gluon propagator, can be only of dynamical origin. The only dynamical mechanism in QCD which can produce such configurations in the vacuum, is the self-interaction of massless gluons – the main dynamical NL effect in QCD. Hence, the above-mentioned mass gap appears on dynamical ground. Let us remind that precisely this self-interaction in the UV limit leads to AF.

We have explicitly shown that the low-frequency components of the virtual fields in the true vacuum should have larger amplitudes than those of a PT ("bare") vacuum [10], indeed. "But it is to just this violent IR behavior that we must look for the key to the low energy and large distance hadron phenomena. In particular, the absence of quarks and other colored objects can only be understood in terms of the IR divergences in the self-energy of a color bearing objects" [33]. So, let us introduce the following definitions:

(i). The power-type IR singularity which is more severe than the exact power-type IR singularity of the free gluon propagator will be called a severe (or equivalently NP IR) singularity. In other words, the NP IR singularity is more severe than $1/q^2$ at $q^2 \rightarrow 0$.

(ii). At the same time, the IR singularity which is as much singular as the exact power-type IR singularity of the free gluon propagator, i.e., as much singular as $1/q^2$ at $q^2 \rightarrow 0$, will be called PT IR singularity.

It makes worth emphasizing in advance that the decomposition of the full gluon propagator into the INP and PT parts (2.42) can be made exact (see below). From the distribution theory point of view the NP IR singularities defined above present a rather broad and important class of functions with algebraic singularities [24]. This explicit derivation shows how precisely the NP IR singularities, accompanied with a mass gap in the corresponding powers, may appear in the vacuum of QCD. Thus, the NP IR singularities should be summarized (accumulated) into the full gluon propagator and effectively correctly described by its structure in the deep IR domain, presented by its INP part. The second step is, of course, to assign a mathematical meaning to the integrals, where such kind of the NP IR singularities will explicitly appear, i.e., to define them correctly in the IR region (see the next Sec.).

One important thing should be made perfectly clear. In the exact calculation of a separate diagram the dependence on the characteristic masses (determining the deviation of the full gluon propagator from the free one in the IR and UV regions) is hidden. In other words, these masses cannot be "seen" by the calculation of the finite number of diagrams, which may be not even gauge-invariant. An infinite number of the corresponding diagrams should be summed up in order to trace such NP masses (i.e., to go beyond PT). The final result of such summation should, in principle, be gauge-invariant. In the weak coupling regime we know how to do this with the help of the renormalization group equations. As a result, the dependence on $\Lambda_{QCD} \equiv \Lambda_{PT}$ will finally appear. At the same time, we do not know how to solve these equations in the strong coupling regime. So, in order to avoid this problem, we decided to show the existence of a mass gap explicitly, by extracting the deep IR asymptotics of the gluon propagator within the separate relevant diagrams. The rest of the problem is to sum up an infinite number of the most singular (leading and next-to-leading) contributions in order to see whether or not a mass gap will finally survive. Precisely this program will be carried out in what follows. We will show that a mass gap remains, indeed, and consequently the full gluon propagator becomes unavoidably severely singular in the deep IR domain.

E. Discussion

1. A necessary generalization

The three-gluon proper vertex vanishes when all the gluon independent momenta involved go to zero, i.e., $T_3(0,0) \rightarrow T_3^0(0,0) = 0$. This is also true for the ghost-gluon proper vertex when all the momenta involved go to zero, namely $G_\mu(0,0) = 0$. Because of this behavior, the three-gluon and ghost-gluon vertices should not play any noticeable role in the deep IR structure of the full gluon propagator. Though separate terms of an infinite series presented by the corresponding skeleton loop integrals (2.7) (this loop integral does not exhibit any IR singularities at all), (2.8), (2.15) and (2.16) may be formally singular (and hence depend on the mass gap), their general tensor decompositions are to be present by the decompositions (2.9) and (2.10) but with different invariant functions for the skeleton loop integrals (2.15) and (2.16), of course. In other words, these terms will not survive after summing up an infinite number of the corresponding contributions. At the same time the four-gluon proper vertex is not zero when all the gluon momenta involved go to zero, i.e., $T_4(0,0,0) \rightarrow T_4^0(0,0,0) \neq 0$. This is the main dynamical source of the additional IR singularities (and hence of the mass gap), which are hidden in the corresponding two-loop skeleton integral (2.17). That is why it plays so important role in the IR structure of the full gluon propagator.

One can conclude that for all the skeleton loop integrals mentioned above the exact $q = 0$ limit is smooth. Thus, not loosing generality, they produce the contributions which are of the order $O(q^2)$ always. However, this is not the case for the two-loop skeleton integral (2.17), which contains the four-gluon vertices. In this case the exact $q = 0$ limit is singular, and the next-to-leading constant contributions are multiplied by the divergent quantities. This everything means that the tensor decomposition of the NL part $T_g[D](q) \equiv T_{\mu\nu}^g[D](q)$ is necessarily to be generalized as follows:

$$T_{\mu\nu}^g[D](q) = \delta_{\mu\nu} \left[\frac{\Delta^4}{q^2} L_g^{(1)}(q^2) + \Delta^2 L_g^{(2)}(q^2) + q^2 T_g^{(3)}(q^2) \right] + q_\mu q_\nu \left[\frac{\Delta^2}{q^2} L_g^{(4)}(q^2) + T_g^{(5)}(q^2) \right], \quad (2.43)$$

where $T_g^{(n)}(q^2)$ at $n = 3, 5$ are invariant dimensionless functions. They are regular functions of q^2 , i.e., they can be present by the corresponding Taylor expansions, but possessing AF at infinity, and depending thus on Λ_{QCD} in this limit. They are saturated by the skeleton loop integrals (2.15), (2.16) and (2.17). At the same time, the invariant dimensionless functions $L_g^{(n)}(q^2)$ at $n = 1, 2, 4$ are to be present by the corresponding Laurent expansions, namely

$$L_g^{(1,2,4)}(q^2) \equiv L_g^{(1,2,4)}(q^2, \Delta^2) = \sum_{k=0}^{\infty} (\Delta^2/q^2)^k a_k^{(1,2,4)}, \quad (2.44)$$

where the numbers $a_k^{(1,2,4)}$ by themselves are expansions in the coupling constant squared (see below). These invariant functions are to be saturated by the skeleton loop integral (2.17) only. Let us emphasize the inevitable appearance of the mass gap Δ^2 . It characterizes the nontrivial dynamics in the IR region. This precisely makes the difference between the linear and NL insertions into the gluon self-energy. Evidently, this difference is due to different dynamics: in the linear part there is no explicit direct interaction between massless gluons, while in the NL part there is. When the mass gap is zero then this decomposition takes the standard form. So, the generalization (2.43) makes the explicit dependence on the mass gap of the full gluon propagator perfectly clear. Let us emphasize once more that ghost and quark degrees of freedom contribute into the INP part of the full gluon propagator numerically only, and as functions they contribute into its PT part. Neither ghost nor quark skeleton loops (2.7) and (2.8), which appear in Eq. (2.4), can cancel its severely singular behavior in the IR, which was demonstrated above. It was due to the pure YM part, i.e., to its NL part of the gluon SD equation (2.5). More precisely mainly to its two-loop skeleton integral, which contain the four-gluon vertices only.

2. The role of ghosts

It is well known that in order to maintain the unitarity of S -matrix in QCD the ghosts have to cancel unphysical degrees of freedom (longitudinal ones) of the gauge bosons [1, 7, 17, 18]. Evidently, this is due to the general decomposition of the ghost skeleton loop (2.9), which shows that it always gives the contribution of the order q^2 . In the iteration solution for the gluon propagator (2.4), $D(q) = D^0(q) - D^0(q)T_{gh}(q)D^0(q) + D^0(q)T_{gh}(q)D^0(q)T_{gh}(q)D^0(q) + \dots$, it cancels one of q^2 in the denominator, which comes from the free gluon propagator. Thus, each term in this expansion becomes always as singular as $1/q^2$. Precisely this makes it possible for ghosts to cancel, in general, the longitudinal component of the full gluon propagator, which is, by definition, as singular as $1/q^2$. From a technical point of view the cancellation can be explicitly demonstrated in the lowest orders of PT in powers of the coupling constant squared (see, for example Ref. [17]). However, this is valid term by term in PT. In other words, in every order of PT the ghosts will cancel unphysical degrees of freedom of gauge bosons, making thus them always transverse. The above-mentioned cancellation in all orders of PT means that it goes beyond PT. It is a general feature, i.e., it does not depend on whether the solution, for example to the gluon SD equation is PT or NP, singular or regular at origin, etc. In other words, the general role of ghosts should not be spoiled by any truncation scheme (approach).

On the other hand, the general decomposition (2.9) of the ghost skeleton loop (2.7) takes place if and only if (iff) the full ghost propagator (Euclidean signature) $G(k) = -(i/k^2[1 + b(k^2)])$, where $b(k^2)$ is the ghost self-energy, is as singular as $1/k^2$ at $k^2 \rightarrow 0$. When the ghost self-energy is zero, i.e., $b(k^2) = 0$, then the full gluon propagator becomes the free one, i.e., $G(k) \rightarrow G_0(k) = -(i/k^2)$. Thus the IR singularity of the full ghost propagator cannot be more severe than the exact IR singularity of the free ghost propagator in order to maintain the cancellation role of unphysical degrees of freedom of gauge bosons by ghosts at any nonzero covariant gauge in all sectors of QCD. There is no way for ghosts to cancel severe IR singularities, which are of dynamical origin due to the self-interaction of massless gluons in the true QCD vacuum. There is no doubt left that the full gluon propagator is essentially severely modified in the IR because of the response of the NP QCD vacuum, which is not provided by the PT vacuum.

However, there exists one gap in these arguments. If one chooses by hand the Landau gauge $\xi = 0$ from the very beginning, then the unphysical longitudinal component of the full gluon propagator vanishes. Only the physical transverse component will contribute to the full gluon propagator, and it may become regular at zero in this case, indeed. Otherwise, it is always singular at the origin because the existence of the longitudinal component always produces, at least, the IR singularity $1/q^2$ (see Eq. (2.1)). In this case there is no restriction on the behavior of the ghost propagator in the IR, and it may become (depending on the truncation scheme) more singular in the IR than its free counterpart. In Ref. [38] (and references therein) precisely this type of the solution (regular gluon propagator and more singular than the free one ghost propagator) to the system of the SD equations in the Landau gauge has been found. However, this solution is due to the choice of the special Landau gauge, so it is a gauge artifact solution. Being thus a gauge artifact, it can be related to none of the physical phenomena such as quark and gluon confinement, SBCS, etc, which are, by definition, manifestly gauge-invariant. At the same time, gauge artifact solutions may exist as formal solutions to the SD system of equations. If a regular at zero gluon propagator will be found in a manifestly gauge-invariant way (i.e., in the way which does not explicitly depend on the particular covariant or non-covariant gauge choice), only then it should be taken seriously into the consideration. To our present knowledge a manifestly gauge-invariant solution for the smooth gluon propagator, which will not compromise the general role of ghosts, is not yet found. Moreover, there exists a serious doubt, in our opinion, that such kind of the solution can be found at all. Thus, we are left with singular at the origin gluon propagator, which is possible in any covariant gauge. In principle, the free gluon propagator can be also used in any gauge. The Feynman gauge free gluon propagator in the IR has been used by Gribov [39] in order to investigate the quark confinement problem within precisely the SD system of dynamical equations approach.

Evidently, the iteration solution (2.38) does not reproduce all aspects of the general iteration solution (2.18). The pure gluon loops are reproduced up to g^2 and g^4 orders, while the ghost and quark loops can be taken into account to all orders in g^2 . Moreover, the most important general feature of the iteration solution, mentioned above is not seen clearly. The main purpose of this paper is to establish the deep IR structure of the full gluon propagator, not solving the gluon SD equation directly, which is a formidable task, anyway. At the same time, we will show the way how to reconstruct the structure of the full gluon propagator in the IR domain in complete agreement with the structure of its general iteration solution (2.18). However, it is convenient first in the next Sec. to emphasize the distribution nature of severe IR singularities.

In summary, we have discussed the general properties of the gluon SD equation and its formal iteration solution. We have explicitly shown how the NP IR singularities inevitably appear in the QCD vacuum. We have discussed the role of ghosts, and it has been also explained why the smooth in the IR gluon propagator is a gauge artifact.

III. IR DIMENSIONAL REGULARIZATION WITHIN THE DISTRIBUTION THEORY

In general, all the Green's functions in QCD are generalized functions, i.e., they are distributions. This is true especially for the NP IR singularities due to the self-interaction of massless gluons in the QCD vacuum. They present a rather broad and important class of functions with algebraic singularities, i.e., functions with nonsummable singularities at isolated points [24] (at zero in our case). Roughly speaking, this means that all relations involving distributions should be considered under corresponding integrals, taking into account the smoothness properties of the corresponding class of test functions (for example, $\varphi(q)$ below. Let us note in advance that in part III we will establish the class of test functions). In principle, any regularization scheme (i.e., how to parameterize severe IR singularities and thereby to put them under control) can be used; it should, however, be compatible with DT [24].

Let us consider the positively definite ($P > 0$) squared (quadratic) Euclidean form $P(q) = q_0^2 + q_1^2 + q_2^2 + \dots + q_{n-1}^2 = q^2$, where n is the number of the components. The generalized function (distribution) $P^\lambda(q)$, where λ is, in general, an arbitrary complex number, is defined as $(P^\lambda, \varphi) = \int_{P>0} P^\lambda(q) \varphi(q) d^n q$. At $Re\lambda \geq 0$ this integral is convergent and is an analytic function of λ . Analytical continuation to the region $Re\lambda < 0$ shows that it has a simple pole at points [24]

$$\lambda = -\frac{n}{2} - k, \quad k = 0, 1, 2, 3, \dots \quad (3.1)$$

In order to actually define the system of the SD equations in the deep IR domain, it is necessary to introduce the IR regularization parameter ϵ , defined as $D = n + 2\epsilon$, $\epsilon \rightarrow 0^+$ within a gauge-invariant DR method [25]. As a result, all the Green's functions and "bare" parameters should be regularized with respect to ϵ (see below), which is to be set to zero at the end of the computations. The structure of the NP IR singularities is then determined (when n is even number) as follows [24]:

$$(q^2)^\lambda = \frac{C_{-1}^{(k)}}{\lambda + (D/2) + k} + \text{finite terms}, \quad (3.2)$$

where the residue is

$$C_{-1}^{(k)} = \frac{\pi^{n/2}}{2^{2k} k! \Gamma((n/2) + k)} \times L^k \delta^n(q) \quad (3.3)$$

with $L = (\partial^2/\partial q_0^2) + (\partial^2/\partial q_1^2) + \dots + (\partial^2/\partial q_{n-1}^2)$.

Thus, the regularization of the NP IR singularities (3.2) is nothing but the so-called Laurent expansion. Let us underline its most remarkable feature. The order of singularity does not depend on λ , n and k . In terms of the IR regularization parameter ϵ , it is always a simple pole $1/\epsilon$. This means that all power terms in Eq. (3.2) will have the same singularity, i.e.,

$$(q^2)^{-\frac{n}{2}-k} = \frac{1}{\epsilon} C_{-1}^{(k)} + \text{finite terms}, \quad \epsilon \rightarrow 0^+, \quad (3.4)$$

where we can put $D = n$ now (i.e., after introducing this expansion). By "finite terms" here and everywhere a number of necessary subtractions under corresponding integrals is understood [24]. However, the residue at a pole

will be drastically changed from one power singularity to another. This means different solutions to the whole system of the SD equations for different set of numbers λ and k . Different solutions mean, in their turn, different vacua. In this picture different vacua are to be labelled by two independent numbers: the exponent λ and k . At a given number of $D(=n)$ the exponent λ is always negative being integer if $D(=n)$ is an even number or fractional if $D(=n)$ is an odd number. The number k is always integer and positive and precisely it determines the corresponding residue at the pole, see Eq. (3.3). It would not be surprising if these numbers were somehow related to the nontrivial topology of the nD QCD vacuum.

Concluding, let us note that the structure of severe IR singularities in Euclidean space is much simpler than in Minkowski space, where kinematical (unphysical) singularities due to the light cone also exist [1, 24, 40]. In this case it is rather difficult to untangle them correctly from the dynamical singularities, the only ones which are important for the calculation of any physical observable. Also the consideration is much more complicated in the configuration space [24]. That is why we always prefer to work in the momentum space (where propagators do not depend explicitly on the number of dimensions) with Euclidean signature. We also prefer to work in the covariant gauges in order to avoid peculiarities of the noncovariant gauges [1, 41], for example how to untangle the gauge pole from the dynamical one.

In summary, first we have emphasized the distribution nature of the NP IR singularities. Secondly, we have explicitly shown how the DR method should be correctly implemented into DT. This makes it possible to put severe IR singularities under firm mathematical control provided by DT itself, complemented by the DR method.

IV. THE GENERAL STRUCTURE OF THE FULL GLUON PROPAGATOR

To say today that QCD is a NP theory is almost a tautology. The problem is how to define it exactly, since we know for sure that QCD has a PT phase as well because of AF. In order to investigate this problem, namely, how to define the NP phase in QCD, it is convenient to begin with the algebraic decomposition of the full gluon form factor in Eq. (2.1) as follows:

$$d(q^2) = d(q^2) - d^{PT}(q^2) + d^{PT}(q^2) = d^{NP}(q^2) + d^{PT}(q^2), \quad (4.1)$$

where, for simplicity, the dependence on the gauge fixing parameter is omitted. In fact, this formal equation represents one unknown function (the full gluon form factor) as an exact sum of the two other unknown functions, which can be always done. So, at this stage there is no approximation made. We would like to let the PT part of this decomposition to be responsible for the known UV asymptotics (since it is fixed by AF) of the full gluon propagator, while the NP part is chosen to be responsible for its unknown yet IR asymptotics. It is worth emphasizing that in realistic models of the full gluon propagator, the NP part reproduces usually correctly its deep IR asymptotics, determining thus the strong intrinsic influence of the IR properties of the theory on its NP dynamics. Evidently, the decomposition (4.1) represents an exact subtraction of the PT contribution at the fundamental gluon level, and consequently both terms in the right-hand-side of Eq. (4.1) are formally determined in the whole momentum range $[0, \infty)$. Let us emphasize that the full gluon form factor $d(q^2)$ being also NP, nevertheless, is "contaminated" by the PT contributions, while $d^{NP}(q^2)$ due to the subtraction (4.1) is free of them, i.e., it is truly NP.

Substituting the decomposition (4.1) into the full gluon propagator (2.1), one obtains

$$D_{\mu\nu}(q) = D_{\mu\nu}^{INP}(q) + D_{\mu\nu}^{PT}(q), \quad (4.2)$$

where

$$D_{\mu\nu}^{INP}(q) = iT_{\mu\nu}(q)d^{NP}(q^2)\frac{1}{q^2} = iT_{\mu\nu}(q)d^{INP}(q^2), \quad (4.3)$$

$$D_{\mu\nu}^{PT}(q) = i\{T_{\mu\nu}(q)d^{PT}(q^2) + \xi L_{\mu\nu}(q)\}\frac{1}{q^2}. \quad (4.4)$$

Let us remind that the superscript "INP" is the short-hand notation for the intrinsically NP phase in QCD. Its definition will be given below in Sec. 7. The exact decomposition (4.2) has a remarkable feature. The explicit gauge dependence of the full gluon propagator is shifted from its INP part to its PT part. In other words, we want the INP part to be always transverse, while leaving the PT part to be of arbitrary gauge. This exact separation will have also

a dynamical ground. It is clear also that the PT part of the full gluon propagator is, by definition, as much singular as the free gluon propagator's power-type IR singularity. This is the first reason why the longitudinal part of the full gluon propagator has been shifted to its PT part (the longitudinal part has the same IR singularity as the free gluon propagator). At the same time, the PT part of the full gluon propagator otherwise remains arbitrary, but preserving AF. Let us also emphasize that ghost and quark degrees of freedom will contribute into its PT part only. As explained above, the corresponding skeleton loop terms in the gluon SD equation produce only $1/q^2$ -type IR singularities in the gluon self-energy.

We want the INP gluon form factor $d^{INP}(q^2)$ to be responsible for the deep IR structure of the full gluon propagator, which is saturated by severe IR singularities. For this aim, it is convenient to introduce the auxiliary INP gluon form factor as follows:

$$d_{\lambda_k}^{INP}(q^2, \Delta^2) = (\Delta^2)^{-\lambda_k-1} (q^2)^{\lambda_k} f_{\lambda_k}(q^2), \quad (4.5)$$

where the exponent λ_k is, in general, an arbitrary complex number with $Re\lambda_k < 0$ (see below). The mass squared parameter Δ^2 (the above-mentioned mass gap) is responsible for the scale of NP dynamics in the IR region in our approach. The functions $f_{\lambda_k}(q^2)$ are, by definition, dimensionless, regular at zero, and otherwise remaining arbitrary, but preserving AF in the UV limit. And finally the number k is a positive integer, i.e., $k = 0, 1, 2, 3, \dots$ (see above). Evidently, a real INP gluon form factor $d^{INP}(q^2)$, which now should depend on the mass gap as well, i.e., $d^{INP}(q^2) \equiv d^{INP}(q^2, \Delta^2)$, is a sum over all auxiliary $d_{\lambda_k}^{INP}(q^2, \Delta^2)$.

However, this is not the whole story yet. Since we are especially interested in the deep IR structure of the full gluon propagator, the arbitrary functions $f_{\lambda_k}(q^2)$ should be also expanded around zero in the form of the Taylor series in powers of q^2 , i.e.,

$$f_{\lambda_k}(q^2) = \sum_{m=0}^{[-\lambda_k]-(n/2)} \frac{(q^2)^m}{m!} f_{\lambda_k}^{(m)}(0) + \sum_{m=[-\lambda_k]-(n/2)+1}^{\infty} \frac{(q^2)^m}{m!} f_{\lambda_k}^{(m)}(0), \quad (4.6)$$

where $[-\lambda_k]$ denotes its integer number and n is the number of the components in the Euclidean squared form q^2 . Also

$$f_{\lambda_k}^{(m)}(0) = \left(d^m f_{\lambda_k}(q^2) / d(q^2)^m \right)_{q^2=0}. \quad (4.7)$$

As a result, we will be left with the finite sum of power terms with an exponent decreasing by unity starting from $-\lambda_k$. All other remaining terms from the Taylor expansion (4.6), starting from the term having already a PT IR singularity (the second sum in Eq. (4.6)), should be shifted to the PT part of the full gluon propagator in Eq. (4.2). The INP part in Eq. (4.5) then becomes

$$d_{\lambda_k}^{INP}(q^2, \Delta^2) = (\Delta^2)^{-\lambda_k-1} (q^2)^{\lambda_k} \sum_{m=0}^{[-\lambda_k]-(n/2)} \frac{(q^2)^m}{m!} f_{\lambda_k}^{(m)}(0), \quad (4.8)$$

while the piece which is to be shifted to the PT part (4.4) of the full gluon propagator (4.2) is as follows:

$$\begin{aligned} d_{(s)\lambda_k}^{INP}(q^2, \Delta^2) &= (\Delta^2)^{-\lambda_k-1} (q^2)^{\lambda_k} \sum_{m=[-\lambda_k]-(n/2)+1}^{\infty} \frac{(q^2)^m}{m!} f_{\lambda_k}^{(m)}(0) \\ &= (\Delta^2)^{-\lambda_k-1} \sum_{m=0}^{\infty} \frac{(q^2)^m}{(m + [-\lambda_k] - (n/2) + 1)!} f_{\lambda}^{(m+[-\lambda_k]-(n/2)+1)}(0), \end{aligned} \quad (4.9)$$

where the subscript "(s)" means "shifted". The above-mentioned sum over all λ_k is also assumed. The important thing here is that the expression (4.9) does not contain the NP (severe) IR divergences with respect to the gluon momentum, indeed.

V. 4D QCD

We are particularly interested in 4D QCD (i.e., $n = 4$), which is a realistic dynamical theory of strong interactions not only at the fundamental quark-gluon level, but at the hadronic level as well [1]. Let us discuss the gluon propagator (4.2) in more detail for QCD itself. On account of the expansion (4.8) and Eq. (3.1) at $n = 4$ with the obvious identification $\lambda_k \equiv \lambda$, its INP part becomes

$$d^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} d_k^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} (\Delta^2)^{1+k} (q^2)^{-2-k} \sum_{m=0}^k \frac{(q^2)^m}{m!} f_k^{(m)}(0), \quad (5.1)$$

and $f_k^{(0)}(0) \equiv f_k(0)$. Obviously, in this case the subscript " λ_k " should be replaced by the subscript " k ", since $\lambda \equiv \lambda_k = -2 - k$, $k = 0, 1, 2, 3, \dots$. Thus, $d^{INP}(q^2, \Delta^2)$ describes the true (physical) NP vacuum of QCD, while $d_k^{INP}(q^2, \Delta^2)$ describe auxiliary ones, and the former is an infinite sum of the latter ones. The expansion (5.1) is obviously the Laurent expansion in the inverse powers of the gluon momentum squared, which every term ends at the simplest NP IR singularity $(q^2)^{-2}$. The only physical quantity (apart from the mass gap, of course) which can appear in this expansion is the coupling constant squared in the corresponding powers. In QCD it is dimensionless and is evidently included into the f_k functions. However, let us note in advance that all the finite numerical factors and constants (for example, the coupling constant) play no independent role in the presence of a mass gap.

It is instructive to show explicitly expansions for a few first different $d_k^{INP}(q^2, \Delta^2)$, namely

$$\begin{aligned} d_0^{INP}(q^2, \Delta^2) &= \Delta^2 f_0(0) (q^2)^{-2}, \\ d_1^{INP}(q^2, \Delta^2) &= (\Delta^2)^2 f_1(0) (q^2)^{-3} + (\Delta^2)^2 f_1^{(1)}(0) (q^2)^{-2}, \\ d_2^{INP}(q^2, \Delta^2) &= (\Delta^2)^3 f_2(0) (q^2)^{-4} + (\Delta^2)^3 f_2^{(1)}(0) (q^2)^{-3} + \frac{1}{2} (\Delta^2)^3 f_2^{(2)}(0) (q^2)^{-2}, \\ d_3^{INP}(q^2, \Delta^2) &= (\Delta^2)^4 f_3(0) (q^2)^{-5} + (\Delta^2)^4 f_3^{(1)}(0) (q^2)^{-4} + \frac{1}{2} (\Delta^2)^4 f_3^{(2)}(0) (q^2)^{-3} + \frac{1}{6} (\Delta^2)^4 f_3^{(3)}(0) (q^2)^{-2}, \end{aligned} \quad (5.2)$$

and so on. Apparently, there is no way that such kind of an infinite series could be summed up into the finite functions, for example functions which could be regular at zero. That is why the above-mentioned smooth gluon propagator is, in general, very unlikely to exist (see remarks below as well). At the same time, an infinite series (5.1), on account of the relations (5.2), correctly reproduces the algebraic structure of the general iteration solution (2.18), on account of the relations (2.19). Each previous iteration is embodied into the subsequent one with different coefficients. Let us remind now that because of Eq. (4.7) all $f_k^{(m)}(0)$ have the dimensions of the inverse mass squared in powers of m , i.e.,

$$[f_k^{(m)}(0)] = [\Delta^{-2}]^m = [\Delta^2]^{-m}, \quad (5.3)$$

not losing generality. Let also note in advance that the simplest NP IR singularity $(q^2)^{-2}$ is present in each iteration, which emphasizes its special and important role (see below).

Evidently, the expansion (5.1), on account of the relations (5.2), can be equivalently written down as follows:

$$d^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} (q^2)^{-2-k} \sum_{m=0}^{\infty} \frac{1}{m!} (\Delta^2)^{k+m+1} f_{k+m}^{(m)}(0) = \sum_{k=0}^{\infty} (q^2)^{-2-k} (\Delta^2)^{k+1} \sum_{m=0}^{\infty} \frac{1}{m!} \varphi_{k,m}(0), \quad (5.4)$$

where we use the relation

$$f_{k+m}^{(m)}(0) = (\Delta^2)^{-m} \varphi_{k,m}(0), \quad (5.5)$$

which obviously follows from the relation (5.3). Here $\varphi_{k,m}(0)$ are dimensionless quantities, by definition. This expansion explicitly shows that the coefficient at each NP IR singularity is an infinite series itself. It also shows that we can analyze the IR properties of the INP part of the full gluon form factor in terms of the mass gap Δ^2 and the

dimensionless quantities $\varphi_{k,m}(0)$ only, which is very convenient from a technical point of view (see below). This form of the Laurent expansion shows also clearly the dynamical context of the INP part of the full gluon propagator.

As underlined above, the piece of the NP part of the full gluon propagator, which does not suffer from the NP IR singularities with respect to the gluon momentum (the second sum in Eq. (4.6)), should be shifted to its PT part. In QCD it comes from the expression (4.9) and looks like

$$d_{(s)}^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} d_{(s)k}^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} (\Delta^2)^{1+k} (q^2)^{-2-k} \sum_{m=k+1}^{\infty} \frac{(q^2)^m}{m!} f_k^{(m)}(0). \quad (5.6)$$

Obviously, in the replacement

$$d^{PT}(q^2) \implies d^{PT}(q^2) + \phi(q^2, \Delta^2) = d'^{PT}(q^2), \quad (5.7)$$

the function $\phi(q^2, \Delta^2)$ summed up over k becomes

$$\begin{aligned} \phi(q^2, \Delta^2) &= \sum_{k=0}^{\infty} \phi_k(q^2, \Delta^2) = \sum_{k=0}^{\infty} (\Delta^2)^{1+k} (q^2)^{-1-k} \sum_{m=k+1}^{\infty} \frac{(q^2)^m}{m!} f_k^{(m)}(0) \\ &= \sum_{k=0}^{\infty} (\Delta^2)^{1+k} \sum_{m=0}^{\infty} \frac{(q^2)^m}{(m+k+1)!} f_k^{(m+k+1)}(0), \end{aligned} \quad (5.8)$$

and it is free from the NP IR singularities with respect to the gluon momentum.

It has been already mentioned that an infinite series (5.4) cannot be summed up into the some finite functions, which can be regular at origin. Indeed, let us define the coefficients b_k as follows:

$$b_k = \sum_{m=0}^{\infty} \frac{1}{m!} \varphi_{k,m}(0), \quad k = 0, 1, 2, 3, \dots \quad (5.9)$$

It is also instructive to introduce new dimensionless variables as $x_k = \Delta_k^2/q^2$, where $\Delta_k^2 = c_k \Delta^2$ and $b_k = c_k^k$. All these relations are relevant at $k = 1, 2, 3, \dots$. The Laurent expansion (5.4) then become

$$d^{INP}(q^2, \Delta^2) = \Delta^2 (q^2)^{-2} \left[b_0 + \sum_{k=1}^{\infty} (x_k)^k \right]. \quad (5.10)$$

There is no any hope to find the sum of an infinite series over k even at all $x_k \ll 1$ (though this is not the case, since we are interested in the IR region when q^2 is small). Moreover, in order to get finite result this sum should cancel first an arbitrary constant b_0 and than to cancel the simplest NP IR singularity $(q^2)^{-2}$. Of course, this is unbelievable. In other words, such kind of an infinite series are nonsummable in mathematics due to the arbitrariness of x_k (which, in fact, is reduced to the arbitrariness of the coefficients b_k). The full gluon propagator is therefore inevitably IR singular (no smooth in the IR gluon propagator). The only hope to proceed further is to get rid of this sum by carrying out an appropriate IR renormalization program (see below).

In summary, starting from the preceding section and especially in this section, we have established the deep IR structure of the full gluon propagator in QCD. By construction, it is an infinite sum over all the NP IR singularities, accompanied by the corresponding powers of the mass gap. At the same time, its structure in the PT regime remains arbitrary (but preserving AF).

VI. GENERAL IR RENORMALIZATION PROGRAM

The regularization of the NP IR singularities in QCD is determined by the Laurent expansion (3.4) at $n = 4$ as follows:

$$(q^2)^{-2-k} = \frac{1}{\epsilon} a(k) [\delta^4(q)]^{(k)} + f.t., \quad \epsilon \rightarrow 0^+, \quad (6.1)$$

where $a(k)$ is a finite constant depending only on k and $[\delta^4(q)]^{(k)}$ represents the k th derivative of the δ -function (see Eqs. (3.2) and (3.3)). We point out that after introducing this expansion everywhere one can fix the number of dimensions, i.e., put $D = n = 4$ for QCD without any further problems. Indeed there will be no other severe IR singularities with respect to ϵ as it goes to zero, but those explicitly shown in this expansion. Let us underline that while the initial expansion (5.4) is the Laurent expansion in the inverse powers of the gluon momentum squared, the regularization expansion (6.1) is the Laurent expansion in powers of ϵ . This means that its regular part is as follows: $f.t. = (q^2)^{-2-k} + \epsilon(q^2)^{-2-k} \ln q^2 + O(\epsilon^2)$, where for the unimportant here definition of the functional $(q^2)^{-2-k}$ see Ref. [24]. Let us note that in the Laurent expansion (5.4) there are no $\ln q^2$ -type terms, since they appear in the PT part. At the same time, when the Laurent expansion (5.4) will be dimensionally regularized with the help of the expansion (6.1) for each NP IR singularity, then such kind of terms will appear. However, they will appear in the next-to-leading terms, and therefore will be suppressed in the $\epsilon \rightarrow 0^+$ limit. Thus, as it follows from the Laurent expansion (6.1) that is dimensionally regularized, any power-type NP IR singularity, including the simplest one, scales as $1/\epsilon$ as it goes to zero. Just this plays a crucial role in the IR renormalization of the theory within our approach.

A. IR renormalization of the gluon SD equation

We are able now to consider the IR renormalization of the full gluon SD equation. Fortunately, the gluon SD equation (2.4) does not contain unknown scattering amplitudes, which usually are determined by the infinite series of the multi-loop skeleton diagrams. It is a closed system in the sense that there is a dependence only on the pure gluon vertices, quark- and ghost-gluon vertices and on the corresponding propagators. Its IR renormalization can be carried out with the help of the above-mentioned quantities only. For this purpose and on account of Eq. (2.13), let us rewrite it in the following form, namely

$$D(q) = D^0(q) - D^0(q)T_{gh}(q)D(q) - D^0(q)T_q(q)D(q) + D^0(q)\frac{1}{2}T_t(q)D(q) + D^0(q)\frac{1}{2}T_1(q)D(q) + D^0(q)\frac{1}{2}T_2(q)D(q) + D^0(q)\frac{1}{6}T'_2(q)D(q), \quad (6.2)$$

where all quantities have been explicitly defined in the expressions (2.7), (2.8) and in Eqs. (2.14)-(2.17). For simplicity, here and below we neglect Dirac indices, since they play no any role in tracking down of the IR singularities in the corresponding equations and expressions. The next step is to introduce the IR regularized quantities. In the presence of such severe IR singularities (6.1), all the quantities should, in principle, depend on ϵ as well, i.e., they become IR regularized. So, one has to put formally

$$\begin{aligned} g^2 &= X(\epsilon)\bar{g}^2, \quad G(k) = \tilde{Z}_2(\epsilon)\bar{G}(k), \quad S(p) = Z_2(\epsilon)\bar{S}(p), \\ G_\mu(k, q) &= \tilde{Z}_1(\epsilon)\bar{G}_\mu(k, q), \quad \Gamma_\mu(p, q) = Z_1^{-1}(\epsilon)\bar{\Gamma}_\mu(p, q), \\ D(q) &= Z_d(\epsilon)\bar{D}(q), \\ T_3(q, q_1) &= Z_3(\epsilon)\bar{T}_3(q, q_1), \\ T_4(q, q_1, q_2) &= Z_4(\epsilon)\bar{T}_4(q, q_1, q_2). \end{aligned} \quad (6.3)$$

In all these relations the quantities with an overbar are, by definition, IR renormalized, i.e., they are supposed to exist as ϵ goes to zero. In both quantities, the IR regularized and IR renormalized ones, the explicit dependence on ϵ is omitted, for simplicity. In the corresponding IR multiplicative renormalization (IRMR) constants this dependence is not omitted in order to distinguish them clearly from the corresponding UVMR constants. Since we are interested in the IR renormalization of the SD equation for the full gluon propagator, it is convenient not to distinguish between the IR renormalization of its INP and PT parts at this stage. Substituting these relations into the gluon SD equation (6.2), one obtains

$$\begin{aligned} \bar{D}(q) &= D^0(q) - D^0(q)\bar{T}_{gh}(q)\bar{D}(q) - D^0(q)\bar{T}_q(q)\bar{D}(q) + D^0(q)\frac{1}{2}\bar{T}_t(q)\bar{D}(q) \\ &+ D^0(q)\frac{1}{2}\bar{T}_1(q)\bar{D}(q) + D^0(q)\frac{1}{2}\bar{T}_2(q)\bar{D}(q) + D^0(q)\frac{1}{6}\bar{T}'_2(q)\bar{D}(q). \end{aligned} \quad (6.4)$$

The gluon SD equation (6.4) will exactly reproduce the structure of Eq. (6.2) iff the following IR convergence conditions hold:

$$\begin{aligned}
Z_d(\epsilon) &= 1, & X(\epsilon)Z_d^2(\epsilon) &= 1, \\
X(\epsilon)Z_d^3(\epsilon)Z_3(\epsilon) &= 1, & X^2(\epsilon)Z_d^5(\epsilon)Z_3^2(\epsilon) &= 1, & X^2(\epsilon)Z_d^4(\epsilon)Z_4(\epsilon) &= 1, \\
X(\epsilon)Z_2^2(\epsilon)Z_1^{-1}(\epsilon)Z_d(\epsilon) &= 1, & X(\epsilon)\tilde{Z}_2^2(\epsilon)\tilde{Z}_1(\epsilon)Z_d(\epsilon) &= 1.
\end{aligned} \tag{6.5}$$

In connection with this system a few remarks are in order. Here and everywhere, one can show that all the finite but arbitrary and different constants, which only ones can appear in the right-hand-side of these relations, can be put to unity not losing generality. Moreover, this is general feature of our approach. In all the IR convergence conditions, all the finite, but arbitrary numbers can be put to unity, by simply redefining the corresponding IRMR constants as well as the corresponding IR renormalized quantities. In what follows this always will be assumed. These IR convergence conditions should be fulfilled simultaneously and independently, of course, in order to maintain the algebraic and tensor structure of the gluon SD equation itself. This makes it possible not to lose even one bit of the information on the QCD vacuum, the dynamical and topological structures of which are supposed to be reflected by the solutions of this equation. Let us also note in advance that two last IR convergence conditions will be known as quark and ghost self-energy IR convergence conditions, respectively (part II).

Evidently, the solutions of these relations are

$$Z_d(\epsilon) = X(\epsilon) = 1, \quad Z_3(\epsilon) = Z_4(\epsilon) = 1, \quad Z_2^2(\epsilon)Z_1^{-1}(\epsilon) = 1, \quad \tilde{Z}_2^2(\epsilon)\tilde{Z}_1(\epsilon) = 1. \tag{6.6}$$

Thus, the IRMR constants of quark and ghost degrees of freedom remain undetermined at this stage. They will be determined via the corresponding Slavnov-Taylor (ST) identities [1, 2, 3, 4, 6, 7, 8, 9, 12], which relate them to each other. It is worth mentioning in advance that they also are IR finite from the very beginning, i.e., $Z_2(\epsilon) = Z_1^{-1}(\epsilon) = \tilde{Z}_2(\epsilon) = \tilde{Z}_1(\epsilon) = 1$ (see part II). However, the most important observation here is that $X(\epsilon) = 1$, which means that the QCD coupling constant is IR finite from the very beginning as well, i.e., $g^2 = \bar{g}^2$. That is $Z_d(\epsilon) = 1$ is in complete agreement with our general result that by the IR renormalization of a mass gap only, we are able to remove all severe IR singularities from the INP part of the full gluon propagator (see the next Subsec.). Moreover, from the fact that the full gluon propagator is not IR renormalized ($Z_d(\epsilon) = 1$), it is easy to understand that the gauge fixing parameter is also IR finite from the very beginning, i.e., $\xi = \bar{\xi}$. Let us underline also that from $Z_d(\epsilon) = X(\epsilon) = 1$ follows that $Z_3(\epsilon) = Z_4(\epsilon) = 1$, and not vice versa. Evidently, one can start from any place in the SD system of equations, for example to start from the quark and ghost sectors, ST identities, etc. However, finally the system of the corresponding IR convergence conditions will have the same solutions (6.6), of course.

B. IR renormalization of a mass gap

In the preceding Subsec. it has been proven that the QCD coupling constant and the gauge fixing parameter are not IR renormalized. They are IR finite from the very beginning, i.e., $g^2 = \bar{g}^2$ and $\xi = \bar{\xi}$. In this case it is convenient to rewrite the Laurent expansion (5.4) for the INP part of the full gluon propagator as follows:

$$d^{INP}(q^2, \Delta^2) = \sum_{k=0}^{\infty} (q^2)^{-2-k} (\Delta^2)^{k+1} \sum_{m=0}^{\infty} a_{k,m}(\xi) g^{2m}, \tag{6.7}$$

i. e., to show the dependence on the coupling constant squared and the gauge fixing parameter explicitly. The rest in the arbitrary numbers $a_{k,m}(\xi)$ is simply the product of the numerical factors like π 's in different powers, eigenvalues of the color group generators (we are not considering the numbers of different colors and flavors as free parameters of the theory), etc. Let us also remind that these numbers contain ghost and quark degrees of freedom in all orders of linear PT, which, nevertheless, have been integrated out numerically, so some of these numbers are UV divergent.

Similar to the relations (6.3), let us now introduce the IR renormalized mass gap

$$\Delta^2 = X_\Delta(\epsilon) \bar{\Delta}^2, \tag{6.8}$$

where $X_\Delta(\epsilon)$ is the corresponding IRMR constant. We already know that all the NP IR singularities, which can appear in the full gluon propagator scale as $1/\epsilon$ with respect to ϵ (see Eq. (6.1)). Introducing now the so-called IR convergence conditions as follows:

$$X_\Delta^{k+1}(\epsilon) = \epsilon \bar{A}_k(\epsilon), \quad k = 0, 1, 2, 3, \dots, \quad \epsilon \rightarrow 0^+, \tag{6.9}$$

the cancellation of the NP IR singularities with respect to ϵ will be guaranteed term by term (since the NP IR singularities are completely independent distributions) in the Laurent expansion (6.7). Here $\bar{A}_k(\epsilon)$ are the IR renormalized quantities, which, by definition, exist (are not singular) as ϵ goes to zero. In terms of the IR renormalized quantities the INP part of the full gluon propagator (6.7) then becomes

$$d^{INP}(q^2, \bar{\Delta}^2) = \epsilon \sum_{k=0}^{\infty} (\bar{\Delta}^2)^{1+k} (q^2)^{-2-k} \bar{B}_k(\epsilon), \quad (6.10)$$

where

$$\bar{B}_k(\epsilon) = \bar{A}_k(\epsilon) a_k = \bar{A}_k(\epsilon) \sum_{m=0}^{\infty} a_{k,m}(\xi) g^{2m}, \quad (6.11)$$

and it exists as ϵ goes to zero (at any $k = 0, 1, 2, 3, \dots$).

Due to the distribution nature of the NP IR singularities, in principle, two different cases should be considered.

I. There is an explicit integration over the gluon momentum, then the gluon form factor (6.10) becomes

$$d^{INP}(q^2, \bar{\Delta}^2) = \sum_{k=0}^{\infty} (\bar{\Delta}^2)^{1+k} a(k) [\delta^4(q)]^{(k)} \bar{B}_k(\epsilon), \quad (6.12)$$

provided the INP part of the full gluon propagator to be IR finite from the very beginning, i.e., its IRMR constant will not depend on ϵ at all as it goes to zero. It is easy to understand that in this case we have to replace the NP IR singularity $(q^2)^{-2-k}$ by its Laurent expansion (6.1), which shows that $(q^2)^{-2-k}$ always scales as $1/\epsilon$. The so-called *f.t.* shown there become terms of the order ϵ as $\epsilon \rightarrow 0^+$, so they vanish in this limit.

II. There is no explicit integration over the gluon momentum, then the INP part of the full gluon form factor (6.10) vanishes, i.e.,

$$d^{INP}(q^2, \bar{\Delta}^2) \sim \epsilon, \quad \epsilon \rightarrow 0^+. \quad (6.13)$$

In this case the NP IR singularities $(q^2)^{-2-k}$ cannot be treated as distributions, and therefore the Laurent expansion (6.1) is not to be used, i.e., the above-mentioned functions $(q^2)^{-2-k}$ are to be considered as standard mathematical functions. This behavior can be treated as gluon confinement criterion (see below). It simply means that there are no transverse gluons in the IR, i.e., at large distances one cannot detect gluons as free particles.

Let us emphasize now that the IR convergence conditions (6.9) should be valid at any k , in particular at $k = 0$, then from Eq. (6.9) it follows

$$X_{\Delta}(\epsilon) = \epsilon \bar{A}_0, \quad \epsilon \rightarrow 0^+, \quad (6.14)$$

where we put $\bar{A}_0 \equiv \bar{A}_0(0)$, i.e., its value in the $\epsilon \rightarrow 0^+$ limit. Thus the mass gap is IR renormalized as follows:

$$\Delta^2 = \epsilon \bar{\Delta}^2, \quad \epsilon \rightarrow 0^+, \quad (6.15)$$

where we include an arbitrary but finite constant \bar{A}_0 into the IR renormalized mass gap $\bar{\Delta}^2$, and retaining, for simplicity, the same notation. This means that in what follows we can put it to unity, not losing generality, i.e., $\bar{A}_0 = 1$. From the IR convergence conditions (6.9) it follows that $\bar{A}_k(\epsilon) \sim \epsilon^k$, which through the relation (6.11) yields $\bar{B}_k(\epsilon) \sim \epsilon^k$ as well. In other words, the solution obtained at $k = 0$ should be used in the IR convergence condition at $k = 1$ and so on.

It is instructive to rewrite the Laurent expansion (6.10), which is already IR renormalized, as follows:

$$d^{INP}(q^2, \bar{\Delta}^2) = \epsilon \bar{\Delta}^2 (q^2)^{-2} \sum_{m=0}^{\infty} a_{0,m}(\xi) g^{2m} + \epsilon \bar{\Delta}^2 (q^2)^{-2} \sum_{k=1}^{\infty} (\bar{\Delta}^2/q^2)^k \bar{B}_k(\epsilon). \quad (6.16)$$

Since we already know that the quantities $\bar{B}_k(\epsilon)$ scale as ϵ^k , then the second sum in this decomposition is additionally suppressed in the $\epsilon \rightarrow 0^+$ limit (it scales as ϵ^2 as $\epsilon \rightarrow 0^+$, at least). We are thus left with the first term in this expansion, namely

$$d^{INP}(q^2, \bar{\Delta}^2) = \epsilon \bar{\Delta}^2 (q^2)^{-2} \sum_{m=0}^{\infty} a_m(\xi) g^{2m}, \quad (6.17)$$

where $a_m(\xi) \equiv a_{0,m}(\xi)$. In other words, in the Laurent expansion (6.16) only the term which contain the simplest NP IR singularity with respect to the gluon momentum $(q^2)^{-2}$ will survive as $\epsilon \rightarrow 0^+$.

In summary, by the IR renormalization of the mass gap Δ^2 only, we can remove all the NP IR singularities, parameterized in terms of the IR regularization parameter ϵ , from the INP part of the gluon propagator. In its turn, this makes it possible to fix the functional dependence of the INP part of the full gluon propagator.

VII. ZMME QUANTUM MODEL OF THE TRUE QCD GROUND STATE

The true QCD ground state is a very complicated confining medium, containing many types of gluon field configurations, components, ingredients and objects of different nature [1, 42, 43, 44, 45, 46]. Its dynamical and topological complexity means that its structure can be organized at both the quantum and classical levels. It is definitely "contaminated" by such gluon field excitations and fluctuations, which are of PT origin, nature and magnitude. Moreover, it may contain such extra gluon field configurations, which cannot be detected as possible solutions to the QCD dynamical equations of motion, either quantum or classical, for example vortex-type ones [47]. The only well known classical component of the QCD ground state is the topologically nontrivial instanton-antiinstanton type of fluctuations of gluon fields, which are solutions to the Euclidean YM classical equations of motion in the weak coupling regime [48, 49]. However, they are by no means dominant but, nevertheless, playing a special role in the QCD vacuum. In our opinion their main task is to prevent quarks and gluons to freely propagate in the QCD vacuum. It seems to us that this role does not contradict their standard interpretation as an tunneling trajectories linking vacua with different topology [1, 49] (and references therein). A highly nontrivial dynamical and topological structure of the QCD vacuum emerges within our approach (for more detail qualitative discussion see Ref. [12]).

Today there is no doubt left that dynamical mechanisms of the important NP quantum phenomena such as color confinement and SBCS are closely related to the above-mentioned complicated and topologically nontrivial structure of the QCD vacuum. On the other hand, it also becomes clear from what was discussed above that the NP IR singularities play an important role in the large distances behavior of QCD. For that very reason, any correct NP model of color confinement and SBCS necessary turns out to be a realistic model of the true QCD vacuum and the other way around. Evidently, our approach to it is based on the importance of such gluon fields which are solutions to the QCD quantum equations of motion.

Our quantum, dynamical model of the true QCD ground state is based on the existence and the importance of such kind of the NP excitations and fluctuations of gluon fields which are precisely due to the self-interaction of massless gluons only without explicitly involving some extra degrees of freedom. They are to be summarized (accumulated) into the purely transverse part of the full gluon propagator, and are to be effectively correctly described by its severely singular structure in the deep IR domain. We will call them the purely transverse singular gluon fields, for simplicity. At the microscopic, dynamical level the self-interaction of massless gluons is a twofold: the three- and four-gluon vertices. We have established that just the latter ones are behind the purely transverse singular gluon fields.

At this stage, it is difficult to identify actually which type of gauge field configurations can be finally formed by the purely transverse singular gluon fields in the QCD ground state, i.e., to identify relevant field configurations: chromomagnetic, self-dual, stochastic, etc. However, if these gauge field configurations can be absorbed into the gluon propagator (i.e., if they can be considered as solutions to the corresponding SD equation), then its severe IR singular behavior is a common feature for all of them. Being thus a general phenomenon, the existence and the importance of quantum excitations and fluctuations of severely IR degrees of freedom inevitably lead to the general ZMME effect in the QCD ground state. That is why we call our model of the QCD ground state as the ZMME quantum model, or simply zero modes enhancement (ZME, since we work always in the momentum space). For preliminary investigation of this model see our papers [50, 51] and references therein.

Our approach to the QCD true ground state, based on the general ZMME phenomenon there, can be analytically formulated in terms of the exact decomposition of the full gluon propagator as follows:

$$D_{\mu\nu}(q) = D_{\mu\nu}^{INP}(q, \Delta^2) + D_{\mu\nu}^{PT}(q), \quad (7.1)$$

where the INP part of the full gluon propagator effectively becomes

$$D_{\mu\nu}^{INP}(q, \Delta^2) = iT_{\mu\nu}(q)d^{INP}(q^2, \Delta^2) = iT_{\mu\nu}(q) \times \Delta^2(q^2)^{-2} \sum_{m=0}^{\infty} a_m(\xi)g^{2m}, \quad (7.2)$$

since only the first term in the expansion (6.16) will finally survive in the $\epsilon \rightarrow 0^+$ limit, as explained above. The PT part of the full gluon propagator $D_{\mu\nu}^{PT}(q)$ in any case remains undetermined. Anyway, it is not important in our approach (see below). Let us only remind that it depends on a new gluon PT form factor $d^{PT}(q^2)$ (see Eq. (5.7)). At the level of a single full gluon propagator, its PT part is defined as to be of the arbitrary covariant gauge and it is as much singular as $1/q^2$ in the IR.

A. Confinement criterion for gluons

Before going to the direct solution of the gluon SD equation within our approach, it is worth discussing the properties of the obtained solution for the full gluon propagator in more detail. We already know that the full gluon propagator is not IR renormalized, i. e., it is IR finite from the very beginning $D(q) = \bar{D}(q)$ ($Z_d(\epsilon) = 1$). As mentioned above, however, the two different cases should be considered due to the distribution nature of the simplest NP IR singularity $(q^2)^{-2}$, which saturates the INP part of the full gluon propagator.

I. If there is an explicit integration over the gluon momentum (the so-called virtual gluon due to Mandelstam [10]), then from Eq. (7.2) it follows

$$D_{\mu\nu}^{INP}(q, \bar{\Delta}^2) = iT_{\mu\nu}(q)\bar{\Delta}^2\pi^2\delta^4(q), \quad (7.3)$$

i.e., in this case we have to replace the NP IR singularity $(q^2)^{-2}$ in Eq. (7.2) by its δ -type regularization (6.1) at $k = 0$. Also we should always take into account the relation (6.15) for the IR renormalization of the mass gap in order to express everything in terms of the IR renormalized quantities. Thus, in this case the auxiliary IRMR constant for the INP part of the full gluon propagator, defined as $D^{INP}(q) = Z_d^{INP}(\epsilon)\bar{D}^{INP}(q)$, is equal to unity, i.e.,

$$Z_d^{INP}(\epsilon) = \epsilon/\epsilon = 1, \quad \epsilon \rightarrow 0^+. \quad (7.4)$$

The δ -type regularization is valid even for the multi-loop skeleton diagrams, where the number of independent loops is equal to the number of the gluon propagators. In the multi-loop skeleton diagrams, where these numbers do not coincide (for example, in the diagrams containing three or four-gluon proper vertices), the general regularization (6.1) should be used (i.e., the derivatives of the δ -functions), and not the product of the δ -functions at the same point, which has no mathematical meaning in the DT sense [24] (for concrete examples see appendix A). At the same time, the auxiliary IRMR constant for the INP part (7.4) remains the same, of course. The so-called "f.t." terms in the Laurent expansion (6.1) become terms of the order of ϵ , at least, so they vanish in the $\epsilon \rightarrow 0^+$ limit. In Eq. (7.3) an infinite series over the coupling constant squared is included to the IR renormalized mass gap $\bar{\Delta}^2$ with retaining the same notation, for simplicity, making it thus UV renormalized as well (let us remind that some coefficients in this expansion are UV divergent). This also makes it possible to eliminate the explicit dependence on the gauge fixing parameter in the gluon form factor for the INP part of the full gluon propagator.

II. If there is no explicit integration over the gluon momentum (the so-called actual gluon [10]), then the function $(q^2)^{-2}$ in Eq. (7.2) cannot be treated as the distribution. The INP part of the full gluon propagator in this case disappears as ϵ as $\epsilon \rightarrow 0^+$, namely

$$D_{\mu\nu}^{INP}(q, \bar{\Delta}^2) \sim \epsilon, \quad \epsilon \rightarrow 0^+, \quad (7.5)$$

i.e., the IR renormalization of the mass gap (6.15) only comes out into the play. This means that any amplitude (more precisely its INP part, see the next Subsec. B) for any number of soft-gluon emissions (no integration over their momenta) will vanish in the IR limit in our picture. In other words, there are no transverse gluons in the IR, i.e., at large distances (small momenta) there is no possibility to observe gluons experimentally as free particles. Thus, the color gluons can never be isolated. This behavior can be treated as the gluon confinement criterion (see also Ref. [9]), and it supports the consistency of the exact solution (7.2) for the INP part of the full gluon propagator. Evidently, this behavior does not explicitly depend on the gauge choice in the full gluon propagator, i.e., it is a manifestly

gauge-invariant as it should be, in principle. These two observations greatly simplify the analysis of the gluon SD equation within our approach (see below). Thus, in this case the auxiliary IRMR constant for the INP part of the full gluon propagator vanishes, i.e.,

$$Z_d^{INP}(\epsilon) = \epsilon, \quad \epsilon \rightarrow 0^+, \quad (7.6)$$

so that the full gluon propagator is reduced to the PT one in the $\epsilon \rightarrow 0^+$ limit.

It makes sense to postpone the comparison of our manifestly gauge-invariant criterion for gluon confinement (7.5) with the color confinement criterion due to Kugo and Ojima (KO) [52], which is based on ghost degrees of freedom, until part III of our investigation. However, the surprising agreement with the color confinement criterion due to Nishijima and Oehme (NO) [53] (see also Ref. [54]) is to be briefly discussed here. The NO color confinement criterion is formulated as follows:

$$Z_3^{-1} = 0, \quad (7.7)$$

where Z_3^{-1} is, in fact, the renormalization constant of the transverse part of the full gluon propagator. Within our notations it should be identified with the IR renormalization constant of the purely transverse part (i.e., the INP part) of the full gluon propagator. From Eq. (7.6) in the final $\epsilon \rightarrow 0^+$ limit, one then obtains

$$Z_d^{INP}(0) \equiv Z_d^{INP} = 0, \quad (7.8)$$

in complete agreement with the NO criterion (7.7). Behind our criterion (7.5) is clear dynamical mechanism, while the NO criterion (7.7) is rather kinematical (metric confinement) than dynamical. However, in both cases the transverse (physical) gluons are removed from the spectrum in a quite similar way. The coincidence between the IRMR and UVMR constants of the full gluon propagator once more may signify deep properties of the full theory [30]. We think that this coincidence is neither accidental nor formal, and it deserves to be investigated in more detail elsewhere.

B. INP phase in QCD

For the sake of self-consistency and transparency of our approach to low-energy QCD, it is convenient to discuss in more detail what we mean by the INP phase in QCD. In the decomposition (7.1) $D_{\mu\nu}^{PT}(q)$ is given by Eq. (4.4), on account of the replacement for the PT gluon form factor in Eq. (5.7). The INP part of the full gluon propagator is, in general, given in Eqs. (7.3) and (7.5) for the above-discussed two different cases. Let us remind that within our approach all severe IR singularities of the dynamical origin possible in QCD are to be incorporated into the full gluon propagator and are to be effectively correctly described by its INP part. So, all other QCD proper vertices can be considered as regular functions with respect to all the gluon momenta involved. In QCD there is some kind of a correspondence between the pure gluon proper vertices in the deep IR region (i.e., when all the gluon momenta involved go to zero) and their point-like counterparts, namely $T_4(0,0,0) \rightarrow T_4^0 \neq 0$, while $T_3(0,0) \rightarrow T_3^0(0,0) = 0$. If, nevertheless, they might be singular, then this correspondence is violated and it would require a completely different investigation, anyway.

In principle, all other fundamental quantities in QCD could be formally decomposed similar to the decomposition (7.1) for the full gluon propagator. Evidently, this should be done for quantities, which explicitly depend on the gluon momenta, i.e., proper vertices. The decomposition does not make any sense for the coupling constant, quark and ghost propagators, since they do not depend on the gluon momentum. This is also true for the quark- and ghost-gluon proper vertices, since they depend on the quark and ghost momenta, which are completely independent from the gluon momentum, playing the role of the momentum transfer in these vertices. The only proper vertices which makes sense to decompose are the pure gluon vertices, since they crucially depend on all the gluon momenta involved. However, we define the INP phase in QCD in more general terms, which includes the decomposition of the full gluon propagator only as follows:

- (i) It is always transverse, i.e., it depends only on physical degrees of freedom of gauge bosons.
- (ii) Before the IR renormalization, the presence of the NP IR singularities $(q^2)^{-2-k}$, $k = 0, 1, 2, 3, \dots$ is only possible.
- (iii). After the IR renormalization, the INP part of the full gluon propagator is fully saturated by the simplest NP IR singularity, and all other NP IR singularities will be additionally suppressed in the $\epsilon \rightarrow 0^+$ limit.
- (iv). There is an inevitable dependence on the mass gap Δ^2 , so that when it formally goes to zero, then the INP phase vanishes, while the PT phase survives.

Evidently, this definition implies that the INP part of any multi-loop skeleton diagram in QCD should contain only the INP parts of all the corresponding gluon propagators. At the same time, the PT part of any multi-loop skeleton diagram always remains of arbitrary gauge. It may even contain the terms, where the NP IR singularities are present along with the PT IR ones as well (the so-called general PT term, see discussion below). The difference between the NP IR singularities $(q^2)^{-2-k}$ and the PT IR singularity $(q^2)^{-1}$ is that the latter is not defined by its own Laurent expansion (6.1) that is dimensionally regularized like the former ones. That is why it does not require the IR renormalization program itself.

C. A few comments

We already know that when there is no explicit integration over the gluon momentum, then the full gluon propagator is reduced to the PT one. In other words, in this case $D = Z_d^{PT}(\epsilon)\bar{D}^{PT} = D^{PT}$, by definition, where $Z_d^{PT}(\epsilon)$ is the auxiliary IRMR constant of the PT part of the full gluon propagator. However, from the general solution $Z_d = 1$ ($D = \bar{D}$) it follows that $Z_d^{PT} = Z_d = 1$ and hence $D^{PT} = \bar{D}^{PT}$ (all the IRMR constants, which do not depend on ϵ , become an arbitrary, but finite numerical constants, and they can be put to unity not losing generality, as mentioned above). Let us remind that the gauge fixing parameter is not IR renormalized as well. In principle, this is not surprising, since the PT part of the full gluon propagator is automatically free from the NP IR singularities, by definition. At the same time, the IR finiteness of the full gluon propagator has been achieved (constructed) by the nontrivial IR renormalization procedure. It includes the IR renormalization of the required mass gap by taking into account the distribution nature of the NP IR singularities (see above).

From QCD sum rules it is well known that AF is stopped by power-type terms reflecting the growth of the coupling in the IR. Approaching the deep IR region from above, the IR sensitive contributions were parameterized in terms of a few quantities (gluon and quark condensates, etc.), while direct access to NP effects (i.e., to the deep IR region) was blocked by the IR divergences [55, 56]. Our approach to NP QCD, in particular, to its true ground state within the just formulated ZMME model is a further step into the deep IR region (in fact, we are deeply inside it), since DT allows one to correctly deal with the NP IR singularities.

Evidently, the ZMME mechanism for quark confinement is nothing but the well forgotten IRS one, which can be equivalently referred to as a strong coupling regime [1, 32]. Indeed, at the very beginning of QCD it was expressed a general idea [32, 33, 34, 35, 36] that the quantum excitations of the IR degrees of freedom, because of the self-interaction of massless gluons in the QCD vacuum, made it only possible to understand confinement, DCSB and other NP effects. In other words, the importance of the deep IR structure of the true QCD vacuum has been emphasized as well as its relevance to quark confinement, DCSB, etc., and the other way around. This development was stopped by the wide-spread wrong opinion that severe IR singularities cannot be put under control. We have explicitly shown that the correct mathematical theory of quantum YM physical theory is the theory of distributions (the theory of generalized functions) [24, 57], complemented by the DR method [25]. They provide a correct treatment of these severe IR singularities without any problems. Thus we come back to the old idea but on a new basis that is why it becomes new ("new is well forgotten old"). In other words, we put the IRS mechanism of quark confinement on a firm mathematical ground provided by DT. Moreover, we also emphasize the role of the purely transverse gauge fields in this mechanism. After the authors of Ref. [35] we can repeat that what we want eventually is not AF but transverse IRS/ZMME.

In the light of the above-mentioned correspondence between the pure gluon proper vertices in the deep IR region and their point-like counterparts, it becomes almost clear that in order to control firmly the deep IR region within our approach, we need precisely the point-like pure gluon vertices rather than their proper counterparts. Since the INP part of the full gluon propagator is finally saturated by the simplest NP IR singularity $(q^2)^{-2}$ only, this implies all the pure gluon proper vertices to become effectively point-like ones. Otherwise, each simplest NP IR singularity multiplied by the corresponding gluon momentum coming from the proper vertex effectively becomes less singular, and therefore is to be shifted to the corresponding PT parts in accordance with our method. This is just happening in the case of the three-gluon vertex (it depends, at least, linearly on all the gluon momenta involved). At the same time, the four-gluon vertex remains the source of the NP IR singularities in the gluon propagator. The quark- and ghost-gluon proper vertices at zero momentum transfer (i.e., at zero gluon momentum) are to be accounted for as well. However, as mentioned above, we will always operate with the full vertices, since summing up an infinite number of the diagrams, for example with point-like pure gluon vertices, we again come to their proper counterparts. The decomposition of the full gluon propagator under DT, complemented by the DR method, is only necessary and completely enough to firmly control the IR region in QCD within our approach.

Let us make one thing perfectly clear. The important observation was that due to the distribution nature of the NP IR singularities, two different types of the IR renormalization of the INP part of the full gluon propagator have been required. The principal distinction between them is whether there is an explicit integration over the gluon

momentum or not, preserving, nevertheless, the IR finiteness of the full gluon propagator ($Z_d(\epsilon) = 1$). The pure gluon vertices crucially depend on all the gluon momenta involved. So, it will be necessary to distinguish between their IR renormalization whether there is an explicit integration over all the gluon propagators or there is an explicit integration not over all ones (associated in both cases with each pure gluon vertex). The previous result $Z_3(\epsilon) = Z_4(\epsilon) = 1$ should be preserved as well (see part II). Also let us note that calculating, for example the one-gluon exchange potential between heavy quarks, $d^{INP}(q^2, \Delta^2) \sim \Delta^2(q^2)^{-2}$ should be used, by including an infinite sum into the mass gap. Finally this combination becomes a string tension.

Working always in the momentum space, we are speaking about the purely transverse singular gluon fields responsible for color confinement in our approach. Discussing the relevant field configurations, we always will mean the functional (configuration) space. Speaking about relevant field configurations (chromomagnetic, self-dual, stochastic, etc), we mean, of course, the low-frequency modes of all of these virtual transverse fields. Only large scale amplitudes of these fields ("large transverse gluon fields") are to be taken into account by the INP part of the full gluon propagators. All other frequencies are to be taken into account by corresponding PT part of the gluon propagators. To speak about specific field configurations that are solely responsible for color confinement is not the case, indeed. The low-frequency components/large scale amplitudes of all the possible in the QCD vacuum the purely transverse virtual fields are important for the dynamical and topological formation of such gluon field configurations which are responsible for color confinement and other NP effects within our approach to low-energy QCD. For convenience, we will call them the purely transverse severely singular gluon field configurations.

In summary, we have formulated the ZMME model of the true QCD ground state in terms of the gluon propagator. On one hand, this makes it possible to establish its structure in the IR region. On the other hand, it allows one to formulate the gluon confinement criterion in Eq. (7.5) in a manifestly gauge-invariant way. In the same way, we have defined the INP phase in QCD in terms of the corresponding decomposition of the full gluon propagator only. It is worth emphasizing, that just the INP phase will be responsible for the NP effects in QCD, such as quark confinement, DBCS, etc., within our approach to low-energy QCD.

VIII. SOLUTION OF THE GLUON SD EQUATION WITHIN OUR APPROACH

As already said, the gluon SD equation (2.5) does not contain unknown scattering amplitudes, so it can be directly solved within our approach. By direct solution, we mean that the INP part of the full gluon propagator, as it is determined in Eqs. (7.3) and (7.5), identically satisfies the corresponding INP part of the gluon SD equation. Substituting the decomposition (7.1) into the gluon SD equation (2.5), one obtains

$$D^{INP}(q) + D^{PT}(q) = \tilde{D}^0(q) + \tilde{D}^0(q)T_g[D](q)D^{INP}(q) + \tilde{D}^0(q)T_g[D](q)D^{PT}(q). \quad (8.1)$$

In this equation and in what follows, we omit the explicit dependence on the Dirac indices, playing no any role in tracking down of the IR singularities in the corresponding integrals. The nonlinear part of the gluon self-energy $T_g[D](q)$ should be also decomposed as follows:

$$T_g[D](q) = T_g[D^{INP} + D^{PT}](q) = T_g^{INP}[D^{INP}](q) + T_g^{PT}[D](q), \quad (8.2)$$

so that the INP part will depend on D^{INP} only, while the PT part remains arbitrary, i.e., it may depend on both D^{INP} and D^{PT} , as mentioned above. Separating now between the INP and PT parts in the gluon SD equation (8.1), on account of the decomposition (8.2), one obtains

$$D^{INP}(q) - \tilde{D}^0(q)T_g^{INP}[D^{INP}](q)D^{INP}(q) - \tilde{D}^0(q)T_g^{PT}[D](q)D^{INP}(q) = -D^{PT}(q) + \tilde{D}^0(q) + \tilde{D}^0(q)T_g[D](q)D^{PT}(q). \quad (8.3)$$

We already know that the INP part of the full gluon propagator with momentum q vanishes as ϵ (see Eq. (7.5)), since there is no explicit integration over the gluon momentum q . So, if the INP part of the nonlinear part of the gluon self-energy $T_g^{INP}[D^{INP}](q)$ and the composition $T_g^{PT}[D](q)$ (i.e., $T_g[D](q)$ itself) do not produce any problems in the $\epsilon \rightarrow 0^+$ limit, then the left-hand-side of Eq. (8.3) identically vanishes in this limit. We will be left therefore with the gluon SD equation for the PT part of the full gluon propagator as follows:

$$D^{PT}(q) = \tilde{D}^0(q) + \tilde{D}^0(q)T_g[D](q)D^{PT}(q) = \tilde{D}^0(q) + \tilde{D}^0(q)\left[T_g^{INP}[D^{INP}](q) + T_g^{PT}[D](q)\right]D^{PT}(q). \quad (8.4)$$

The main purpose of this Sec. and appendix A is to show explicitly that this is so, indeed.

The nonlinear gluon self-energy $\Sigma_g^{NL}(q) = T_g[D](q)$ is the sum of the four terms due to Eq. (2.13). Let us begin with Eq. (2.14), which describes the so-called tadpole term. Substituting the exact decomposition (7.1) for the full gluon propagator, one obtains

$$T_t = T_t^{INP} + T_t^{PT}, \quad (8.5)$$

where

$$T_t^{INP} = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 D^{INP}(q_1) = -\bar{\Delta}_t^2 T_4^0 \quad (8.6)$$

and the PT part is given in the appendix A. Here and everywhere below, all the finite constants will be included into the corresponding IR finite mass scale parameters (here into $\bar{\Delta}_t^2$). For the INP part has been used the relation (7.5), since there is an explicit integration over the gluon momentum q_1 . So, the tadpole term does not produce problems as $\epsilon \rightarrow 0^+$.

On account of the decompositions (7.1), the integral (2.15) becomes

$$T_1(q) = T_1^{INP}(q) + T_1^{PT}(q), \quad (8.7)$$

where

$$T_1^{INP}(q) = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_3^0(q, -q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{INP}(q_1) D^{INP}(q - q_1), \quad (8.8)$$

and the PT part is a sum of the three terms, which are shown in the appendix A. Integrating over the gluon momentum q_1 in this integral with the help of Eq. (7.3), one obtains

$$T_1^{INP}(q) = -\bar{\Delta}_1^2 T_3^0(q, 0, -q) T_3(-q, 0, q) D^{INP}(q) \sim \epsilon, \quad \epsilon \rightarrow 0^+, \quad (8.9)$$

since there is no explicit integration over the gluon momentum q (see Eq. (7.5)).

Let us now evaluate first Eq. (2.17), since it contains three gluon propagators in comparison with Eq. (2.16), by reminding that in both equations $q_3 = q - q_1 + q_2$. Substituting again the decompositions (7.1), one obtains

$$T_2'(q) = T_2'^{INP}(q) + T_2'^{PT}(q), \quad (8.10)$$

where

$$T_2'^{INP}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{INP}(q_1) D^{INP}(-q_2) D^{INP}(q - q_1 + q_2), \quad (8.11)$$

and the PT part is a sum of the seven terms, which are shown in the appendix A. Integrating over the gluon momenta q_1 and q_2 (let us note that, in general, $D(q) = D(-q)$) with the help of Eq. (7.3), one obtains

$$T_2'^{INP}(q) = \bar{\Delta}_2^4 T_4^0 T_4(-q, 0, 0, q) D^{INP}(q) \sim \epsilon, \quad \epsilon \rightarrow 0^+, \quad (8.12)$$

since in this equation there is no explicit integration over the gluon momentum q (see Eq. (7.5)).

Let us now evaluate Eq. (2.16). Again substituting the decompositions (7.1), one obtains

$$T_2(q) = T_2^{INP}(q) + T_2^{PT}(q), \quad (8.13)$$

where

$$T_2^{INP}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{INP}(q_1) D^{INP}(-q_2) D^{INP}(q - q_1 + q_2) D^{INP}(q - q_1), \quad (8.14)$$

and the PT part is a sum of the fifteen terms, which are shown in the appendix A. Integrating over the gluon momenta q_1 and q_2 with the help of Eq. (7.3), yields

$$T_2^{INP}(q) = \bar{\Delta}_2^4 T_4^0 T_3(0, q, -q) T_3(-q, 0, q) D^{INP}(q) D^{INP}(q) \sim \epsilon^2, \quad \epsilon \rightarrow 0^+, \quad (8.15)$$

since in this equation there is no explicit integration over the gluon momentum q (see Eq. (7.5)).

The INP part of the nonlinear pure gluon part is a sum of the four terms, namely

$$T_g^{INP}[D^{INP}](q) = \frac{1}{2} T_t^{INP} + \frac{1}{2} T_1^{INP}(q) + \frac{1}{2} T_2^{INP}(q) + \frac{1}{6} T_2'^{INP}(q), \quad (8.16)$$

where each term is given by Eqs. (8.6), (8.9), (8.12) and (8.15). Because of Eq. (8.6) it is finite in the $\epsilon \rightarrow 0^+$ limit,

$$T_g^{INP}[D^{INP}](q) = -\frac{1}{2} \bar{\Delta}_t^2 T_4^0 + O(\epsilon), \quad \epsilon \rightarrow 0^+. \quad (8.17)$$

So, the combination

$$\tilde{D}^0(q) T_g^{INP}[D^{INP}](q) D^{INP}(q) \sim \epsilon, \quad \epsilon \rightarrow 0^+, \quad (8.18)$$

indeed. At the same time, in the appendix A it is shown that the composition $T_g^{PT}[D](q)$ does not produce any problems in the $\epsilon \rightarrow 0^+$ limit. This means that the combination

$$\tilde{D}^0(q) T_g^{PT}[D](q) D^{INP}(q) \sim \epsilon, \quad \epsilon \rightarrow 0^+, \quad (8.19)$$

as well. Thus, we are left with Eq. (8.4) for the PT part of the full gluon propagator only, indeed. This means that the INP part of the full gluon propagator is completely decoupled from the rest of the gluon SD equation, i.e., it identically and independently satisfies the INP part of the gluon SD equation (the left-hand-side of Eq. (8.3)). Eq. (8.4) for the PT part of the full gluon propagator can be further simplified. On account of Eq. (8.17), it becomes

$$D^{PT}(q) = \tilde{D}^0(q) + \tilde{D}^0(q) \left[-\frac{1}{2} \bar{\Delta}_t^2 T_4^0 + T_g^{PT}[D](q) \right] D^{PT}(q). \quad (8.20)$$

In the appendix A it is shown that it produces no explicit problems in the $\epsilon \rightarrow 0^+$ limit. However, let us underline once more that it is not our problem, since we are, in principle, not responsible for the PT phase in QCD.

Concluding this Sec., let us clarify the terminology. The general gluon SD equation (2.5) can be written down for the inverse of the full gluon propagator as

$$D^{-1}(q) = [\tilde{D}^0(q)]^{-1} - T_g[D](q) = [\tilde{D}^0(q)]^{-1} - \Sigma_g^{NL}(q), \quad (8.21)$$

so we can call $T_g[D](q) = \Sigma_g^{NL}(q)$ as the NL part of the gluon self-energy. If one begins with the free gluon propagator $D^0(q)$, then the gluon SD equation looks more conventionally

$$D^{-1}(q) = [D^0(q)]^{-1} - \Sigma_g(q), \quad (8.22)$$

where

$$\Sigma_g(q) = \Sigma_{gh}(q) + \Sigma_q(q) + \Sigma_g^{NL}(q), \quad (8.23)$$

and $\Sigma_{gh}(q)$ and $\Sigma_q(q)$ are given by the integrals (2.7) and (2.8), respectively.

IX. GENERAL DISCUSSION

Let us begin our discussion with reemphasizing a few important points.

The first point is that the only place where the most important problem of theoretical particle/nuclear physics – color confinement (together with other NP effects) can be solved is the SD system of dynamical equations of motion, since it contains the full dynamical information (and even more than that) on QCD. To solve this system means to solve QCD itself and vice versa.

The second point is that AF clearly indicates the existence of the NP phase (with its own characteristic scale parameter) in the full QCD. Apparently, this reflects the fact that QCD as a whole is UV stable, and thus IR unstable theory (i.e, it has no IR stable fixed point, indeed) [1, 32]. In other words, QCD as a whole is the PT UV and NP IR divergent theory, but, nevertheless, it is IR renormalizable as well.

The third point is that the only place where the NP dynamics can be introduced is the deep IR region, since the PT structure of QCD is controlled by AF.

The fourth point is that the deep IR region is dominated (saturated) by the self-interaction of massless gluons in the true QCD vacuum (no Schwinger-Higgs phase transition in QCD, so the $SU(3)$ color gauge symmetry is exact and gluons remain massless [1]). The above-mentioned NL interaction leads to the ZMME in the gluon propagation, and not to its effective/dynamical mass.

The fifth point is that at the microscopic level we identify unambiguously the fundamental four-gluon interaction as the main dynamical source which naturally leads to the enhancement of the zero momentum degrees of freedom in the gluon propagation (self-energy). So, severe IR singularities are introduced not by hand. The corresponding purely transverse severely singular gluon field configurations are intrinsically peculiar to the true QCD ground state.

The sixth point is that at level of the gluon SD equation the main source of the above-mentioned IR instability of QCD is the two-loop skeleton term, which contains the four-gluon vertices only. Precisely its additional IR singularities when all the gluon momenta involved go to zero gives birth to the ZMME effect in the true QCD vacuum.

The seventh point is that any deviation in the behavior of the full gluon propagator from the free one in the IR requires an automatic introduction of a characteristic mass scale parameter responsible for the nontrivial dynamics in the IR domain, the so-called mass gap. It is worth emphasizing that it cannot be interpreted as the effective/dynamical gluon mass, which always remains massless in our approach.

The eight point is that a mass gap appears on dynamical ground. It gains contributions from all powers of the coupling constant squared. So, its determination goes beyond PT, i.e., its physical meaning is essentially NP. However, the coupling constant squared itself plays no any role in the presence of a mass gap.

There is no doubt left that the singular configurations of gluon fields play a permanently dominant role in the large scale quantum structure of the QCD vacuum. Whether they can be "seen" (detected) by other methods or approaches is not so important, since we hope that our general consideration, to which we were restricted here, has been convincing enough. However, there already exist a lot of direct and indirect evidences in favor of the $(q^2)^{-2}$ behavior of the full gluon propagator in the IR domain. Let us mention only a few of them, which are the most important in our opinion:

a). After the pioneering papers of Mandelstam in the covariant (Landau) gauge [10] and Baker, Ball and Zachariasen in the axial gauge [58], the consistency of the $(q^2)^{-2}$ IR singular asymptotics of the full gluon propagator in different gauges with the direct solution of the gluon SD equation has been repeatedly confirmed (see, for example Refs. [11, 12, 59, 60, 61, 62] and references therein). **It is worth emphasizing, however, that Eq. (7.2) is an exact result, and thus it expresses not only the deep IR asymptotics of the full gluon propagator.**

b). The cluster property of the Wightman functions in QCD fails, and this allows such a singular behavior for the full gluon propagator in the deep IR domain [63].

c). Such singular behavior of the full gluon propagator in the IR domain leads to the area law for heavy (static) quarks (indicative of confinement) within the Wilson loop approach [64].

d). Moreover, let us underline that without the $(q^2)^{-2}$ IR singular component in the decomposition of the full gluon propagator in the continuous theory, it is impossible to "see" linearly rising potential between heavy quarks by lattice QCD simulations [65], not involving some extra (besides gluons and quarks) degrees of freedom. Evidently, the above-mentioned smooth gluon propagator cannot provide a linear rising potential between heavy quarks "seen" by lattice simulations.

e). There exists also direct lattice evidence that the zero momentum modes are enhanced in the full gluon propagator (and hence in the effective coupling), indeed [66] (and references therein).

f). A NP finite-size scaling technique was used in Ref. [67] to study the evolution of the running coupling (which, in principle, can be identified with the exact gluon form factor) in the $SU(3)$ YM lattice theory. By using the two-loop β -function it is shown to evolve according to PT at high energies, while at low energies it is shown to grow. Though we do not know the NP β -function yet, nevertheless, this growing tendency has to be acknowledged (see also Ref. [68]).

g). Within DT the $(q^2)^{-2}$ singularity is the simplest NP power-type IR singularity in 4D QCD, while the $(q^2)^{-1}$ singularity is the simplest NP power-type IR singularity in 2D QCD, which confines quarks [20, 21, 22]. Though the QCD vacuum is much more complicated medium than its 2D model, nevertheless, the above-mentioned analogy is promising even in the case of the NP dynamics of light quarks.

h). Some classical models of the QCD vacuum also invoke a $(q^2)^{-2}$ behavior of the gluon fields in the IR domain. For example, it appears in the QCD vacuum as a condensation of the color-magnetic monopoles (QCD vacuum is a chromomagnetic superconductor) proposed by Nambu, Mandelstam and 't Hooft and developed by Nair and Rosenzweig (see Ref. [69] and references therein), as well as in the classical mechanism of the confining medium [70] and in the effective theory for the QCD vacuum proposed in Ref. [71].

i). It is also required to derive the linearly rising potential between heavy quarks within the recently proposed renormalization group flow equations approach [72].

j). It has been shown that the IR singular behavior (7.2) directly leads to quark confinement (in a flavor independent way) and SBCS [51, 73] (and references therein, see also part III of our approach).

A. Subtractions

Let us continue our discussion recalling that many important quantities in QCD, such as gluon and quark condensates, topological susceptibility, the Bag constant, etc., are defined beyond PT only [50, 55, 74]. This means that they are determined by such S -matrix elements (correlation functions) from which all types of the PT contributions should be, by definition, subtracted. Within the 2D covariant gauge QCD we have already described all types of the necessary subtractions to be done in order to define correctly such a truly NP quantity as the quark condensate [21]. Let us emphasize that such type of subtractions are inevitable also for the sake of self-consistency. In low-energy QCD there exist relations between different correlation functions, for example the famous Witten-Veneziano (WV) and Gell-Mann-Oakes-Renner (GMOR) formulae. The former [75, 76] relates the pion decay constant and the mass of the η' meson to the topological susceptibility. The latter [55, 77] relates the chiral quark condensate to the pion decay constant and its mass. Defining the topological susceptibility and the quark condensate by the subtraction of all types of the PT contributions, it would be not self-consistent to retain them in the correlation functions determining the pion decay constant and in the expressions for the pion and η' meson masses.

Anyway, our theory for low-energy QCD, which we call INP QCD, will be precisely defined by the subtraction of all types of the PT contributions (the first subtraction has been already done in Eq. (4.1)). At the fundamental quark-gluon (microscopic) level, the first step is to identify the terms in the corresponding SD equations and ST identities, which should survive after the necessary subtractions. Evidently, from the above it follows that those terms will remain only which depend on the INP parts of the full gluon propagators. At the hadronic (macroscopic) level, the second step is to integrate out quark and gluon degrees of freedom ("hadronization"). At this stage, the subtraction aimed at fixing the point at which the PT parts of the corresponding integrals should be subtracted. In part III we will show that in the quark sector the subtraction point should coincide with the constant of integration of the corresponding SD equation for the INP quark propagator. For the YM sector in part V we will determine the subtraction point by the minimization of the INP vacuum energy density (the effective potential in the absence of the external sources [78, 79, 80]) for pure gluon fields.

Our approach to QCD can be symbolically decomposed into two parts (consisting of three terms), namely

$$\begin{aligned} QCD &= INP \times INP \times \dots \times INP + INP \times PT \times INP \times \dots \times PT + PT \times PT \times \dots \times PT \\ &= (INP + GPT) QCD. \end{aligned} \tag{9.1}$$

Evidently, the first term contains only the INP parts of the full gluon propagators, which appear in any multi-loop skeleton diagrams. In the second term the INP parts can be present along with the PT ones, and finally the third term depends only on the corresponding PT parts. The last two terms can be combined into the so-called general PT (GPT) term. It is easy to understand that we can always render the first INP part to be IR finite (the powers of the mass gap, which scales as ϵ , is equal or higher than the powers of the NP IR singularities, which scale as $1/\epsilon$). This is not the case for the second term, where powers of ϵ coming from the IR renormalization of a mass gap may not match the powers of $1/\epsilon$, coming from the corresponding Laurent expansions that are dimensionally regularized (this will be explicitly shown in part II, for convenience). How the PT part should be rendered UV finite is a well known procedure, and it is not our problem. So, the most dangerous term is the second one, and its NP IR singularities cannot be removed by the IR renormalization of a mass gap only. To render it IR finite, some additional condition on the IR renormalization of the four-gluon proper vertex should be imposed (see part II).

The general idea behind all the subtractions is to completely decouple INP QCD from QCD as a whole, i.e., to proceed from QCD to INP QCD by the subtraction of the GPT part in the symbolical Eq. (9.1) as follows:

$$QCD \implies INP\ QCD = QCD - GPT\ QCD. \quad (9.2)$$

In this theory all numerical results will depend on the mass gap and can be expressed in terms of the finite integrals, which, in their turn, will depend only on the constants of integration of the corresponding equations of motion (and current quark masses in the general, nonchiral case). It is worth emphasizing that the subtraction procedure is not only physically well-motivated, as it follows from above. In part III we will show that it is also well justified mathematically from the DT point of view. Evidently, the basic key in this program is the existence of the NP phase in QCD. In part II of our approach, it will be proven that INP QCD and QCD as a whole are IR renormalizable theories, making thus possible to explain and solve the color confinement phenomenon as well as all other NP effects. It is clear now why color confinement was so difficult to explain and to solve before the proof of the IR renormalizability of QCD. In other words, there is a close relationship between this proof and the solution of the confinement problem and vice versa. It is worth emphasizing that by subtracting an infinity from another infinity in order to get a finite result (see Eq. (9.2)), Lorentz invariance will be not violated. Such kind of the subtractions is a standard procedure in any method of renormalization (to render the theory free from the PT UV or NP IR divergences).

Of course, the QCD Green's functions cannot be gauge-invariant because, by definition, all fields are not. This implicit gauge dependence (of the quark propagator as well as all other Green's functions) always exist and cannot, in principle, be eliminated by any means. This is a general feature of all gauge theories such as QED and QCD. Unfortunately, in gauge theories the main problem is not the above-mentioned unavoidable implicit gauge dependence, but the explicit dependence of the Green's functions on the gauge fixing parameter. As explained in the Introduction, in QED this is not so important. It becomes crucially important in QCD due to its non-Abelian character. In order to make gauge bosons transverse we need ghosts. It is well known how the mechanism of the cancellation of unphysical (longitudinal) degrees of freedom of gauge bosons by ghosts works in PT [17, 31]. Though this is a general feature, beyond PT it is technically not well known. It seems to us that distinguishing between transverse and longitudinal components of the gauge bosons on a dynamical ground, we found the way in the correct direction. Subtracting finally the PT contributions, which are always of an arbitrary covariant gauge, while the INP contributions are always transverse, by construction, we were able to formulate INP QCD in a manifestly gauge-invariant way. However, this does not mean that we need no ghosts. The quark-ghost sector contains a very important piece of information on quark degrees of freedom themselves. Precisely this information should be self-consistently taken into account (see part II).

There exists also an additional but very serious argument in favor of the inevitability of the above-discussed subtractions of the PT contributions at all levels in order to fix the gauge of INP QCD. In his pioneering paper [81] Gribov has investigated the quantization problem of non-Abelian gauge theories using the functional integral representation of the generating functional for non-Abelian gauge fields. It has been explicitly shown that the standard Fadeev-Popov (FP) prescription fails to fix the gauge uniquely and therefore should be modified, i.e., it is not enough to eliminate arbitrary degrees of freedom from the theory. In other words, there is an ambiguity in the gauge-fixing of non-Abelian gauge fields (the so-called Gribov ambiguity (uncertainty), which results in Gribov copies and vice versa). To resolve this problem Gribov has explicitly demonstrated that the above-mentioned modification reduces simply to an additional limitation on the integration range in the functional space of non-Abelian gauge fields, which consists in integrating only over the fields for which the FP determinant is positive [81] (introducing thus the so-called Gribov horizon in the functional space, see also Ref. [82]). As emphasized by Gribov, this affects the IR singularities of PT and results in a linear increase of the charge interaction at large distances. The INP part of QCD is a manifestly gauge-invariant, by construction (it depends only on the transverse (physical) degrees of freedom of gauge bosons). All problems with the gauge fixing discovered by Gribov in the functional space should be attributed to the GPT part of QCD within our approach. Subtracting further the GPT part in order to proceed to INP QCD in accordance with our general method (see Eq. (9.2)), we thus will make it free from the gauge-fixing ambiguity in the momentum space. We fix unambiguously the gauge (transverse) of INP QCD. Finally this will lead to the existence of something like Gribov horizon but in the momentum space. We would like to emphasize that the general proposal to subtract all type of the PT contributions is our solution to the problem of Gribov copies. These subtractions become necessary (inevitable) and important in order to make INP QCD free from this problem, which otherwise will plague the dynamics of any essentially NL gauge systems [82].

A few additional remarks on the subtraction of the PT contributions are in order. Let us remind that in lattice QCD [83] such a kind of an equivalent procedure also exists. In order to prepare an ensemble of lattice configurations for the calculation of any NP quantity or to investigate some NP phenomena, the excitations and fluctuations of gluon fields of the PT origin and magnitude should be "washed out" from the vacuum. This goal is usually to be achieved by using "Perfect Actions" [84], "cooling" [85], "cycling" [86], etc. (see also Refs. [42, 43, 44, 45] and references

therein). Evidently, in lattice QCD this is very similar to our method in continuous QCD in order to proceed to INP QCD. However, there exists also a principle difference. As underlined above, in continuous QCD by the subtraction of the PT contributions at all levels, any NP quantity becomes free of the PT "contaminations" for ever, i.e., it remains truly NP. In lattice QCD, one should go to the continuum limit at the final stage, which means the removal of an UV cut-off. So, all numerical lattice results for any NP quantity, somehow become again inevitably "contaminated" by the finite PT contributions (and thus will be plagued by the above-mentioned Gribov uncertainties). There is no way to escape this fundamental difficulty. That is why the criterion of the quark confinement–area law–derived by Wilson in lattice gauge theory [64] becomes not adequate for the continuous theory [87] (Lorentz invariance is restored but confinement is lost [30, 83]). It is worth emphasizing that area law as the quark confinement criterion makes sense only for heavy quarks. The confinement of light quarks and gluons is the *terra incognita* of lattice QCD. As underlined in the Introduction, that is why the color confinement problem of quarks and gluons can be solved only within the SD system of the dynamical equations approach to continuous QCD. The removal of an IR cut-off (the infinite volume limit) should be done exactly after the continuum limit, and not before it. The chiral limit should be taken last, if necessary (however, care is needed because of the chiral log problem). To discuss lattice QCD some other difficulties such as its essentially PT character as a specific regularization scheme and the absence of a mass gap in the below-discussed sense is beyond the scope of the present investigation.

In QCD sum rules the corresponding subtraction should be also done in order to calculate, for example such truly NP quantity as the gluon condensate. While in our approach we should subtract finally the INP part of the full gluon propagator integrated out over the PT region, in QCD sum rules one needs to subtract the PT solution of the full gluon propagator integrated out over the deep IR region, where it certainly fails (see discussion given by Shifman in Ref. [43]). The necessity of the subtraction of the PT part of the effective coupling constant (integrated out) in order to correctly calculate the gluon condensate by analytic method has been explicitly shown in recent paper [88] as well.

Let us emphasize that in the present investigation the only subtraction which has been done so far was the above-mentioned subtraction in Eq. (4.1), defining thus the INP phase in QCD. In order to define INP QCD itself all other subtractions will be done further later on in the next parts of our approach to low-energy QCD. Here, however, it is instructive to emphasize that the dependence on the mass gap in the full gluon propagator (2.1) should be regular one. Evidently, only in this case it can be considered as responsible for the NP phase in the QCD true vacuum. Introducing it explicitly into the full gluon form factor, the subtraction (4.1) can be equivalently written down as follows:

$$d^{NP}(q^2; \Delta^2) = d(q^2; \Delta^2) - d(q^2; \Delta^2 = 0), \quad (9.3)$$

by obviously identifying $d(q^2; \Delta^2 = 0) \equiv d^{PT}(q^2)$. It is interesting to remember that within our approach the NP phase in QCD dies linearly when the corresponding mass gap goes to zero, while AF dies much more slowly as $1/\ln$ when Λ_{QCD}^2 goes formally to zero. Concluding, let us note that in some cases the mass of a particular particle can be treated as the mass gap. For example, in the dual Abelian Higgs model [89, 90] the mass of the dual boson plays the role of a scale responsible for the NP dynamics. The truly NP gluon form factor defined in Eq. (9.3) retains, nevertheless, the IR singularity of the free gluon propagator, i.e., there is no INP phase in this model. As a result its vacuum with string and without string contributions is unstable against quantum corrections [90].

B. A Mass Gap

That the NP structure of the QCD ground state can be reflected by the deep IR structure of the full gluon propagator is one of the important features of our formulation of the INP phase in QCD. That this theory requires an automatical introduction of a characteristic mass scale parameter, responsible for the NP dynamics, the so-called mass gap, is another relevant feature of it. This is especially important, since there is none in the QCD Lagrangian. As mentioned above, in the presence of a finite mass gap the coupling constant becomes to play no any role. Since a mass gap appears on a dynamical ground, this is also a direct evidence for the "dimensional transmutation" [1, 37], which occurs whenever a massless theory acquires masses dynamically. It is a general feature of spontaneous symmetry breaking in field theories.

The relation of our mass gap to the mass gap introduced by Jaffe and Witten (JW) [91] still remains to be understood. Firstly, we naively identified the JW mass gap Δ_{JW} with our Λ_{NP} [92], where Λ_{NP} is the final version of the mass gap Δ (see part II). However, things are apparently not so simple. In QCD there also exists Λ_{PT} ($\equiv \Lambda_{QCD}$), which is responsible for the nontrivial PT dynamics. The relation between these three scales can be symbolically

reproduced as follows:

$$\Lambda_{NP} \xleftarrow[0 \leftarrow M_{IR}]{\infty \leftarrow \alpha_s} \Delta_{JW} \xrightarrow[M_{UV} \rightarrow \infty]{\alpha_s \rightarrow 0} \Lambda_{PT}, \quad (9.4)$$

where α_s is obviously the fine structure coupling constant of strong interactions, while M_{UV} and M_{IR} are the UV and IR cut-offs, respectively. The right-hand-side limit is well known as the weak coupling regime, while the left-hand-side can be regarded as the strong coupling regime. We know how to take the former [1], and we don't know how to take the latter one yet. However, there is no doubt that the final goal of this limit, namely, the mass gap Λ_{NP} exists, and should be the renormalization group invariant in the same way as $\Lambda_{QCD} (\equiv \Lambda_{PT})$ is. Taking the weak coupling limit, a dependence on the number of flavors appears. Apparently, the strong coupling limit is also flavor dependent. At the same time, it is well understood that the fundamental mass gap should come from the quantum YM theory [91], i.e., it should be, in principle, by definition, flavor independent. This means that the JW mass gap Δ_{JW} is the fundamental mass, determining the scale of QCD as a whole. The mass gap Λ_{NP} and Λ_{PT} are responsible for the NP and PT dynamics, determining thus the deviation of the full gluon propagator from the free one in the IR and UV regions, respectively. They are two different "faces" of the same fundamental mass gap Δ_{JW} ("two sides of the same coin"). However, it is necessary to emphasize that the asymptotic scale parameter Λ_{PT} cannot be considered as the mass gap in the sense understood by JW, while Λ_{NP} looks like very similar to Δ_{JW} . In the INP phase of QCD there is no spectrum in the interval $(0, \Lambda_{NP})$, indeed. It is possible to say that in the INP phase of QCD our mass gap Λ_{NP} plays the same role which in QCD as a whole belongs to the JW mass gap Δ_{JW} .

The JW theorem [91] is formulated as follows:

Yang-Mills Existence And Mass Gap: Prove that for any compact simple gauge group G , quantum Yang-Mills theory on R^4 exists and has a mass gap $\Delta > 0$,

then one of our main results obtained in this paper can be formulated in terms of the similar theorem, namely

I. Yang-Mills Existence And Mass Gap: If four-dimensional quantum Yang-Mills theory with compact simple gauge group $G = SU(3)$ exists then it should have a mass gap $\Delta > 0$.

In fact, we have proven the second (physical) part of the JW theorem. The first (mathematical) part is beyond of reach (apparently, it requires a sophisticated mathematics, indeed). Let us outline the main general points of our investigation/proof.

1. We assume that non-Abelian quantum YM theory exists.
 2. It is IR unstable theory (the ZMME effect).
 3. At the microscopic level, the main dynamical source of this instability is the NL fundamental four-gluon interaction.
 4. Precisely it leads to the NP (severe) IR singularities in the full gluon propagator.
 5. In its turn, this automatically requires the existence of a mass gap, so it appears on a dynamical ground.
- We have done even more than that. The rest of our main results can be also formulated as a following theorem:

II. Yang-Mills Existence And Mass Gap: If four-dimensional quantum Yang-Mills theory with compact simple gauge group $G = SU(3)$ exists and exhibits a mass gap $\Delta > 0$ then it confines gluons.

Let us outline the main general points of this part of our investigation/proof.

6. The distribution nature of the NP IR singularities is to be underlined.
7. The use of the corresponding Laurent expansion that is dimensionally regularized in order to control the above-mentioned NP IR singularities.
8. The IR renormalization properties of the mass gap only lead the confinement of gluons.
9. It will explain the quark confinement and SBCS as well.

Of course, our investigation can be given in mathematically standard way (to formulate theorems, the corresponding lemmas, etc.), however, this is not the case here, indeed. At the same time, we would like to emphasize here that our general conclusion that the mass gap in non-Abelian YM theory arises from the quartic potential $(A \wedge A)^2$ in the action is in complete agreement with that expressed by Feynman [93] (see also Ref. [91]).

The ultimate goal of QCD is to calculate all physical parameters, for example hadron masses, as pure numbers times the characteristic scale, the mass gap [30]. Thus, in general, should be $\Lambda_{NP} = \mu \times \Delta_{JW}$ and $\Lambda_{PT} = \nu \times \Delta_{JW}$, where μ and ν are some pure numbers. Obviously, from the above-mentioned relations one can deduce that $\Lambda_{NP} = (\mu/\nu) \times \Lambda_{PT}$. However, such kind of relations do not mean that both quantities are dependent from each other. Such kind of relations, for example between hadron masses, mean only that we know how to calculate them from first principles in terms of the fundamental mass gap. Let us note in advance that in a pure YM theory there are no good physical numbers

(physical observables which can be directly experimentally measured). So, there is a problem how to formulate a well-defined scale-setting scheme in order to determine any mass gap: the JW mass gap Δ_{JW} or our mass gap Λ_{NP} . Evidently, it should be done in a more sophisticated way in the framework of QCD itself [51, 94].

In the next Subsec. it makes sense to discuss some technical aspects of our approach to low-energy QCD.

C. A technical outlook

Let us begin with some tentative remarks on a possible IR renormalizability of INP QCD and QCD itself, which will be proven in part II. There is no doubt that it is due to the DT fundamental result, which requires that any NP IR singularity with respect to momentum in terms of ϵ should always be $1/\epsilon$. And this does not depend on how the IR regularization parameter ϵ has been introduced in a way compatible with DT itself. On the other hand, this fundamental result relates the IR regularization to the number of space-time dimensions (see discussion in Sec. 2 and Refs. [24, 57]), i.e., the so-called "compactification" [91]. It is easy to imagine that otherwise none of the IRMR programs would be possible. In other words, we know the mathematical theory which has to be used - the theory of distributions [24, 57] (apparently, for the first time the distribution nature of the Green's functions in quantum field theory has been recognized and used in Ref. [95]). It provides the basis for the adequate mathematical investigation of a global character of the NP IR divergences. Each one-loop skeleton diagram diverges as $1/\epsilon$. Moreover, each independent loop part of the multi-loop skeleton diagrams will diverge as $1/\epsilon$ as well (see part II). On the other hand, the UV divergences have a local character, and thus should be investigated term by term in the coupling constant squared. Bearing in mind that INP QCD becomes trivially an UV finite theory (after subtractions of the PT contributions at all levels, which contain an UV divergent tails, see the symbolical Eq. (9.2)) and in order not to complicate notations, that is why we have started from the PT unrenormalized Green's functions.

In principle, none of the regularization schemes (how to introduce the IR regularization parameter in order to parameterize the NP IR divergences and thus to put them under control) should be introduced by hand. First of all, it should be well defined. Secondly, it should be compatible with DT [24]. The DR scheme [25] is well defined, and here we have shown how it should be introduced into DT (complemented by the number of subtractions, if necessary). Though the so-called $\pm i\epsilon$ regularization is formally equivalent to the regularization used in our paper (see again Ref. [24]), nevertheless, it is rather inconvenient for practical use. Especially this is true for the gauge-field propagators, which are substantially modified due to the response of the vacuum (the $\pm i\epsilon$ prescription is designated for and is applicable only to the theories with PT vacua, indeed [39, 82]). Other regularization schemes are also available, for example such as analytical regularization used in Ref. [9] or the so-called Speer's regularization [96]. However, they should be compatible with DT, as emphasized above. Anyway, not the regularization is important but DT itself. Whether the theory is IR multiplicative renormalizable or not depends on neither the regularization nor the gauge. Due to the chosen regularization scheme or the gauge only the details of the corresponding IRMR program can be simplified. In other words, if the theory is proven to be IR or UV renormalizable in one gauge, it is IR or UV renormalizable in any other gauge. This is true for the regularization schemes as well.

It is worth emphasizing the difference between QED and QCD. In former theory an electron/quark loop insertion into the photon propagation with momentum q is only possible. As mentioned above in Subsec. D of Sec. 2, the estimate similar to the estimate (2.37) could be formally performed. However, the summation of an infinite series of the most singular terms in the IR will yield final zero, i.e., the mass gap will not survive in QED. That is why an electron loop insertion into the photon propagation with momentum q requires its proportionality to q^2 in the numerator. In the iteration solution for the photon propagator it cancels one of q^2 in the denominator, and the photon always propagates like the free one, i.e., as $1/q^2$ in the IR (even summing up all insertions). The reason is that in QED the cluster property of the Wightman functions forbids a more singular behavior of the full photon propagator in the IR than the behavior of its free photon counterpart. So, in QED the IR singularity of the full photon propagator (2.1) is as much singular as $1/q^2$, i.e., it is always the PT IR singularity. In QCD, contrary to QED, the cluster property of the above-mentioned Wightman functions fails due to the self-interactions of the massless gluons (the so-called Strocchi theorem [69]). This allows a behavior of the full gluon propagator more singular in the IR than the behavior of its free counterpart. That is why the mass gap finally survives after the summation of an infinite series of the most singular terms in the IR. This is precisely the principal dynamical distinction between these two theories. Thus, the estimate like the estimates, which have derived in Subsec. C of Sec. 2, do not exist in QED, which is Abelian gauge theory (no direct interactions between photons), while in QCD they exist, since it is non-Abelian gauge theory (direct interactions between gluons do exist).

There also exists a principal difference in their IR renormalization procedures. Since in QED the IR singularity cannot be as much singular as $1/q^4$ (which is the simplest NP IR singularity possible in 4D gauge theory), Laurent expansion, that is dimensionally regularized for the purpose of the renormalization, becomes useless. Thus the PT IR singularity can be regularized even by hand, introducing, for example, the gluon/photon "mass", which goes to

zero at the end of the computations. In other words, the PT IR singularity needs only the regularization in one place, i.e., all other Green's functions are not affected by the dependence on the regularization "mass". This is the principal difference between the PT and the NP IR singularities. The latter ones additionally need also the nontrivial renormalization program to be done for the whole theory, as it was done for the YM sector in this paper.

Let us underline also that due to the distribution nature of the NP IR singularities, any solution to the gluon SD equation found in closed form (for example, some combination of special functions), has to be presented further by the series of the Laurent expansion, anyway. This makes it possible to put the corresponding integrals and equations in the deep IR domain under firm mathematical control, provided by correctly applying DT, complemented by the DR method, to each term of the expansion. As we already know, the rest of the Laurent expansion (which begins from the PT IR singularity) is to be shifted to the PT part of the full gluon propagator. Thus, the Laurent expansion (6.7) is a general one. It shows any solution to the INP part of the full gluon propagator as an infinite sum of all possible NP IR singularities. The next step is to perform the IR renormalization program (a scaling analysis with respect to the IR regularization parameter). A crucial role in this analysis belongs to the mass gap. This finally makes it possible to fix the IR structure of the full gluon propagator, presented by its INP part in Eq. (7.2). Thus, we have found a solution to the full gluon propagator (7.1) up to its unimportant PT part, not solving directly the gluon SD equation.

The coupling constant in QCD is an effective one. In fact, it is "running", i.e., it is a function of the gluon momentum. The solution of the gluon SD equation (2.5) for the full gluon form factor $d(q^2)$ (which is dimensionless) can be identified with the QCD effective coupling. Because of AF we know its behavior in the deep UV limit. It depends on the asymptotic scale parameter Λ_{PT} [1, 17, 18, 27, 31]. The behavior of the effective coupling in the deep IR limit has been established here (see also our preliminary publication [97]). It depends on its own scale parameter, the mass gap $\bar{\Delta}^2$. In both cases the presence of the corresponding masses indicates the scaling violation. So, for the calculation of any numerical value of the QCD effective coupling, it is necessary to choose the scale, at which it should be done. In this case, it is difficult to assign a physical meaning to any of its numerical values (see also remarks in Refs. [30, 98]), for example to its critical value, at which some NP phenomena may occur. It is much more relevant to speak about the scale at which some physical phenomena become important or even occur. Just this is happening in INP QCD, the numerical results of which depend on the mass gap being the renormalization group invariant. It gains the contributions from all powers of the QCD strong coupling constant squared. Neither any value of it nor it itself plays any role in our approach.

This is also in complete agreement with our definition of the INP phase in QCD. We distinguish between the two phase in QCD by the character of the IR singularities and the presence of the mass gap. We do not use the coupling constant for this purpose. We already know that the INP part can be present by an infinite series in the coupling constant squared. At the same time, the PT part can be responsible for the gluon field configurations, which, in principle, cannot be even described by an infinite series in the coupling constant, for example something like "quantum" instantons. Apparently, the standard definition of the nonperturbativity based on the impossibility to expand in the coupling constant should be abandoned. The primary feature of the nonperturbativity seems to be the dependence on the corresponding mass gap.

In the functional space we should distinguish between the purely transverse severely singular gluon field configurations, which decrease more slowly than $1/r$ (in fact, they increase at large distances r , at least linearly, the so-called "large gluon fields" [81] (see also Ref. [82])), and the gluon field configurations of an arbitrary covariant gauge, which decrease at least as $1/r$. Of course, this separation corresponds to the difference between the NP (severe) and the PT IR singularities defined above in the momentum space. However, the former ones are prevented from growing up to infinity. The above-described IR renormalization renders the amplitudes of these virtual fields finite, but large enough to provide quark confinement by shifting quarks from the mass shell (see part III). Moreover, the amplitudes of the purely transverse severely singular actual gluon field configurations (no explicit integration over the corresponding gluon momenta, i.e., configurations with external gluon legs or equivalently configurations with soft-gluon emissions) are suppressed providing thus the gluon confinement, see Eq. (7.5). This is our response to the Gribov's *dilemma* formulated in Ref. [82] as follows: "the solution of the confinement problem lies not in the understanding of the interaction of "large gluon fields" but instead in the understanding of how the QCD dynamics can be arranged as to prevent the non-Abelian fields from growing real big". Contrary to this, the interaction of just "large gluon fields" leads to color confinement within our approach. Only two different cases of the purely transverse severely singular actual and virtual gluon field configurations should be carefully distinguished due to the distribution nature of the same NP IR singularities for both cases. The Gribov's *dilemma/mystery* can be also formulated as follows: if indeed the NL interaction of "large gluon fields" is responsible for color confinement (and precisely this statement is to be deduced from his paper [81], though finally Gribov pursued another approach [39, 82]), then why we cannot "see" them. This *dilemma/mystery* is nothing else but the first formulation of the gluon confinement problem (see also Ref. [93]). Based on his paper, this problem in some detail has been discussed by Zwanziger in Ref. [99], where the vanishing of zero-momentum lattice Landau and Coulomb gauge gluon propagators have been investigated. Contrary

to this case, first we have shown that the gluon propagator can be only severely singular in the IR. Secondly, this does not prevent to formulate the gluon confinement criterion in a manifestly gauge-invariant way, and thus to resolve finally the gluon confinement mystery. The problem of Gribov copies in lattice QCD has been recently addressed via the computation of the gluon propagator in the Landau gauge [100].

X. CONCLUSIONS

Emphasizing the highly nontrivial structure of the true QCD ground state in the deep IR region, one can conclude:

- 1). The self-interaction of massless gluons (i.e., the NL gluodynamics) is responsible for the large scale structure of the true QCD vacuum.
- 2). It is saturated by the so-called purely transverse severely singular gluon field configurations.
- 3). Precisely these field configurations are behind the ZMME effect in the true QCD vacuum, which is to be taken into account by the deep IR structure of the full gluon propagator.
- 4). The full gluon propagator thus is inevitably more singular in the IR than its free counterpart.
- 5). This requires the existence of a mass gap, which is responsible for the NP dynamics in the QCD vacuum. It appears on dynamical ground due to the self-interaction of massless gluons only. It cannot be interpreted as the effective/dynamical gluon mass.
- 6). We define the NP and the PT IR singularities as more severe than and as much singular as $1/q^2$, respectively, which is the power-type, exact IR singularity of the free gluon propagator. In the functional space the former ones correspond to the purely transverse gluon field configurations which decrease more slowly than $1/r$ at large distances r . The latter ones correspond to the gluon field configurations of an arbitrary covariant gauge which decrease as $1/r$, at least.
- 7). The main dynamical source of the NP (severe) IR singularities in the full gluon propagator is the two-loop skeleton term of the corresponding SD equation, which contains the four-gluon vertices only (Eq. (2.17)).
- 8). We decompose algebraically (i.e., exactly) the full gluon propagator as a sum of its INP and PT parts. We additionally distinguish between them dynamically by the different character of the IR singularities in each part.
- 9). We have established the deep IR structure of the full gluon propagator, reproduced by its INP part, as an infinite sum over all possible NP IR singularities in the expansion (5.1) or equivalently (5.4).
- 10). The next step is to regularize them correctly, i.e., to use the Laurent expansion (6.1) that is dimensionally regularized with respect to the IR regularization parameter ϵ .
- 11). We emphasize once more that the IR renormalization program is based on an important observation that the NP IR singularities $(q^2)^{-2-k}$, being distributions, always scale as $1/\epsilon$, not depending on the power of the singularity k , i.e., $(q^2)^{-2-k} \sim 1/\epsilon$. It is easy to understand that otherwise none of the IR renormalization program in the INP phase of QCD and in QCD itself would be possible.
- 12). The IR renormalization of the initial mass gap is only needed in order to fix uniquely and exactly the IR structure of the full gluon propagator. It is saturated by the simplest NP IR singularity, the famous $(q^2)^{-2}$, see Eq. (7.2).
- 13). This also means that the mass gap after summing up an infinite number of the corresponding diagrams survives, indeed. It gains contributions from all orders of PT in the coupling constant squared, which remains IR finite from the very beginning as well as the gauge fixing parameter.
- 14). The smooth in the IR the full gluon propagator should be ruled out. Only its PT part can be rendered finite at zero due to the special gauge choice (Landau gauge).
- 15). As a functions ghost and quark degrees of freedom contribute into the PT part of the full gluon propagator within our approach. As integrated out in all orders of linear PT (i.e., numerically) they contribute into its INP part as well.
- 16). On this basis, we have formulated the ZMME model of the true QCD ground state. Due to the distribution nature of the NP IR singularities, two different types in the IR renormalization of the INP part of the full gluon propagator are required, preserving, nevertheless, its IR finiteness ($Z_d(\epsilon) = 1$).
- 17). In its turn, this allows one to rigorously prove that color gluons can never be isolated. The gauge-invariant, analytical formulation of the gluon confinement criterion is given in Eq. (7.5).
- 18). In a manifestly gauge-invariant way, we define the INP phase in QCD at the fundamental gluon level. The corresponding decomposition of the full gluon propagator is only needed in order to firmly control the IR region in QCD within our approach (only it contains explicitly the mass gap).
- 19). We have shown that the INP part of the full gluon propagator identically satisfies the corresponding part of the gluon SD equation.

Our general conclusions are:

- I. The NP structure of the true QCD ground state is to be described by the IR structure of the full gluon propagator.

II. It is an infinite sum over all possible NP IR singularities. Due to their distribution nature any solution to the gluon SD equation has to be always present in the form of the corresponding Laurent expansion. It makes it possible to control both the whole expansion as well as each term of it by the correct use of DT.

III. The mass gap responsible for the NP dynamics in the true QCD ground state is required. This is important, since there is none in the QCD Lagrangian.

IV. The mass gap arises from the quartic gluon potential (since it survives when all the gluon momenta involved go to zero, while the triple gluon potential does not). It just makes the full gluon propagator so singular in the IR.

V. In the presence of a mass gap the QCD coupling constant plays no any role.

VI. Complemented by the DR method, DT puts the treatment of the NP IR singularities on a firm mathematical ground. So, there is no place for theoretical uncertainties. The wide-spread opinion that they cannot be controlled is not justified.

VII. We emphasize the importance of the general relation between the IR regularization and the number of space-time dimensions ("compactification"), which is crucial for the general IR renormalization program.

VIII. We would like to stress the importance of the consideration in the Euclidean momentum space, which is automatically free from the kinematical (unphysical) singularities in the gauge-field propagators.

IX. All this makes it possible to fix uniquely the IR structure of the full gluon propagator in QCD, not solving directly the corresponding SD equation itself. Thus, we have exactly established the interaction between quarks (concerning its pure gluon (i.e., NL) contribution up to its unimportant PT part).

X. This somehow astonishingly radical result has been achieved at the expense of the PT part of the full gluon propagator. It remains of arbitrary covariant gauge and its functional dependence cannot be determined. However, as explained above, we are not responsible for the PT phase in QCD.

XI. Collective motion of all the purely transverse *virtual* gluon field configurations with low-frequency components/large scale amplitudes (the purely transverse severely singular virtual gluon field configurations) is solely responsible for the color confinement phenomenon within our approach.

XII. The amplitudes of all the purely transverse severely singular *actual* gluon field configurations are totally suppressed, leading thus to the confinement of gluons (no transverse gluons at large distances).

XIII. The difference between the above-mentioned *virtual* and *actual* gluon field configurations at the microscopic fundamental level should be traced back to the two different types in the IR renormalization of the INP part of the full gluon propagator.

XIV. In its turn, the above-mentioned difference is determined by the IR renormalization properties of the mass gap only, taking into account the distribution nature of the NP IR singularities, which inevitably appear in the full gluon propagator due to the NL character of the gluodynamics.

XV. We unambiguously identify the main source of the IR instability of QCD. It is the four-gluon vertex at the Lagrangian level, and the two-loop skeleton term, which contains only the four-gluon vertices, at the level of the gluon SD equation. Precisely this interaction exhibits an additional IR singularities (and hence the mass gap) in the corresponding loop integrals when all the gluon momenta involved go to zero.

XVI. Color confinement is an IR renormalization effect within our approach. If AF is mainly determined by the three-gluon interactions, then color confinement is mainly due to the four-gluon interactions.

XVII. We would like to emphasize once more a self-consistency and manifestly gauge invariance of our approach to low-energy QCD.

Our main results can be summarized similar to the JW theorem as follows:

Theorem: If four-dimensional quantum Yang-Mills theory with compact simple gauge group $G = SU(3)$ exists then it exhibits a mass gap and confines gluons.

Concluding finally, let us make a few remarks. How to correctly formulate the quark confinement problem was more or less clear from the very beginning of QCD. As mentioned above, the confinement of heavy quarks can be understood in terms of the linear rising potential between them. The confinement of light quarks can be formulated as the absence of the pole-type singularities in the quark Green's functions [73, 101, 102, 103, 104], which can be generalized on heavy quarks as well. At the same time, how to correctly formulate the gluon confinement problem was not clear, and up to these days the confinement of gluons remained rather mysterious. It seems to us that in the present investigation we have correctly formulated the gluon confinement problem, which allowed us to solve it as well. In other words, the above-formulated theorem is proven here (as rigorously as possible in theoretical physics). In its turn, this will allow us to solve the quark confinement problem (the gauge-invariant formulation of the quark confinement criterion will be given in part III of our approach. For preliminary discussion see our paper [20]). A brief description of the most important results obtained in this paper has been already published in Refs. [96, 105]. Though we are not interested in the PT part of the full gluon propagator, nevertheless it is important for high-energy QCD (for a new analytic approach to PT QCD see Ref. [106]).

Acknowledgments

The author would like to thank S. Adler, H. Georgi, K. Nishijima, V.I. Zakharov, H. Fried, M. Faber, A. Ivanov, V.A. Rubakov (and members of his seminar at INR, RAS), D.V. Shirkov, A.A. Slavnov, A.T. Filippov, G.V. Efimov, V.P. Gusynin, Gy. Pocsik, P. Forgács, K. Toth, B. Lukacs, P. Levai, T. Biro, J. Revai, T. Csorgo and especially Gy. Kluge and J. Nyiri for useful correspondence, discussions, remarks, support and help. We are also grateful to H. Ejiri and H. Toki for support and collaboration and useful discussions at the first stage of this investigation. A financial support from HAS-JINR Scientific Collaboration Fund is to be also acknowledged.

APPENDIX A: THE PT PART OF THE GLUON SD EQUATION

In this appendix we evaluate the gluon SD equation (8.19) for the PT part of the full gluon propagator. The PT part of the so-called tadpole term, which is given in Eq. (2.14), is

$$T_t^{PT} = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 D^{PT}(q_1), \quad (A1)$$

so it is finite in the $\epsilon \rightarrow 0^+$ limit. Moreover, in dimensional regularization with $D^{PT} = D^0$ (but not with $D^{PT} = \tilde{D}^0$) it yields simply zero [17]. That is a reason why in the general iteration solution of the gluon SD equation (2.4) the tadpole contribution can be generally discarded, which means the omission of the term depending on $\bar{\Delta}_t^2$ in Eq. (8.20) as well. So, the tadpole term itself is not important, as emphasized above.

The PT part of the three-gluon vertex contribution into nonlinear gluon self-energy comes from Eq. (2.15) and it is a sum of the three terms

$$T_1^{PT}(q) = \sum_{n=1}^3 T_1^{(n)}(q), \quad (A2)$$

where

$$T_1^{(1)}(q) = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_3^0(q, -q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{INP}(q_1) D^{PT}(q - q_1), \quad (A3)$$

$$T_1^{(2)}(q) = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_3^0(q, -q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{INP}(q - q_1), \quad (A4)$$

$$T_1^{(3)}(q) = g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_3^0(q, -q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(q - q_1). \quad (A5)$$

Integrating over the gluon momentum q_1 in Eq. (A3) and over the gluon momentum $q - q_1$ in Eq. (A4) with the help of Eq. (7.3), one finally obtains

$$\begin{aligned} T_1^{PT}(q) = & - \bar{\Delta}_1^2 \left[T_3^0(q, 0, -q) T_3(-q, 0, q) + T_3^0(q, -q, 0) T_3(-q, q, 0) \right] D^{PT}(q) \\ & + g^2 \int \frac{id^4 q_1}{(2\pi)^4} T_3^0(q, -q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(q - q_1). \end{aligned} \quad (A6)$$

One can conclude that one-loop skeleton PT contribution into the gluon self-energy is under control in the $\epsilon \rightarrow 0^+$ limit, i.e., formally it exists as ϵ goes to zero.

The simplest two-loop contribution into the gluon self-energy is determined by Eq. (2.17). Its PT part is a sum of the seven terms

$$T_2'^{PT}(q) = \sum_{n=1}^7 T_2'^{(n)}(q), \quad (A7)$$

where

$$T_2'^{(1)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{INP}(q_1) D^{PT}(-q_2) D^{INP}(q - q_1 + q_2), \quad (A8)$$

$$T_2'^{(2)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{PT}(q_1) D^{INP}(-q_2) D^{INP}(q - q_1 + q_2), \quad (A9)$$

$$T_2'^{(3)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{PT}(q_1) D^{PT}(-q_2) D^{INP}(q - q_1 + q_2), \quad (A10)$$

$$T_2'^{(4)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{INP}(q_1) D^{INP}(-q_2) D^{PT}(q - q_1 + q_2), \quad (A11)$$

$$T_2'^{(5)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{INP}(q_1) D^{PT}(-q_2) D^{PT}(q - q_1 + q_2), \quad (A12)$$

$$T_2'^{(6)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{PT}(q_1) D^{INP}(-q_2) D^{PT}(q - q_1 + q_2), \quad (A13)$$

$$T_2'^{(7)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{PT}(q_1) D^{PT}(-q_2) D^{PT}(q - q_1 + q_2). \quad (A14)$$

Integrating over the gluon momentum q_1 in Eq. (A8) and further over the gluon momentum $q_2 = -q$, one obtains

$$T_2'^{(1)}(q) = \bar{\Delta}_2'^4 T_4^0 T_4(-q, 0, q, 0) D^{PT}(q). \quad (A15)$$

Integrating over the gluon momentum $-q_2$ in Eq. (A9) and further over the gluon momentum $q_1 = q$, one gets

$$T_2'^{(2)}(q) = \bar{\Delta}_2'^4 T_4^0 T_4(-q, q, 0, 0) D^{PT}(q). \quad (A16)$$

Integrating over the gluon momentum $q_2 = q_1 - q$ in Eq. (A10), one arrives at

$$T_2'^{(3)}(q) = -g^4 \bar{\Delta}_2'^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 T_4(-q, q_1, q - q_1, 0) D^{PT}(q_1) D^{PT}(q - q_1). \quad (A17)$$

Integrating over the gluon momenta q_1 and $-q_2$ in Eq. (A11), one obtains

$$T_2'^{(4)}(q) = \bar{\Delta}_2'^4 T_4^0 T_4(-q, 0, 0, q) D^{PT}(q). \quad (A18)$$

Integrating over the gluon momentum q_1 in Eq. (A12), leads to

$$T_2'^{(5)}(q) = -g^4 \bar{\Delta}_2'^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, 0, -q_2, q + q_2) D^{PT}(-q_2) D^{PT}(q + q_2). \quad (A19)$$

Integrating over the gluon momentum $-q_2$ in Eq. (A13), one gets

$$T_2'^{(6)}(q) = -g^4 \bar{\Delta}_2'^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 T_4(-q, q_1, 0, q - q_1) D^{PT}(q_1) D^{PT}(q - q_1), \quad (A20)$$

and the pure PT contribution (A14) remains unchanged. It is convenient replace $q_2 \rightarrow -q_1$ in Eq. (A19), then the total PT contribution becomes

$$\begin{aligned}
T_2'^{PT}(q) &= \bar{\Delta}_2'^4 \left[T_4^0 T_4(-q, 0, q, 0) + T_4^0 T_4(-q, q, 0, 0) + T_4^0 T_4(-q, 0, 0, q) \right] D^{PT}(q) \\
&- \bar{\Delta}_2'^2 \int \frac{id^4 q_1}{(2\pi)^4} \left[T_4^0 T_4(-q, q_1, q - q_1, 0) + T_4^0 T_4(-q, 0, q_1, q - q_1) + T_4^0 T_4(-q, q_1, 0, q - q_1) \right] D^{PT}(q_1) D^{PT}(q - q_1) \\
&+ g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_4(-q, q_1, -q_2, q - q_1 + q_2) D^{PT}(q_1) D^{PT}(-q_2) D^{PT}(q - q_1 + q_2). \tag{A21}
\end{aligned}$$

Again one can conclude that two-loop skeleton contribution into the gluon self-energy due to four-gluon vertices only is under control in the $\epsilon \rightarrow 0^+$ limit. Here the coupling constant g^4 is included into the corresponding mass gap, for convenience apart from the last term, of course.

In connection with the last integral in Eq. (A21) it is worth making a few remarks. There is no explicit integration over variable q , but there are an explicit integrations over variables q_1 and q_2 . In order to estimate the behavior of this integral in the deep IR domain (setting $T_4 = T_4^0$ and using the free gluon propagators in the Feynman gauge instead of the corresponding D^{PT}), one can utilize the method which has been used in Subsec. B of Sec. 2. Omitting some algebra, to leading order one obtains $T_2'^{(7)}(q) \sim (\Delta^4/q^2)$. We already know, however, that the mass gap scales as ϵ as it goes to zero, i.e., $\Delta^2 = \epsilon \bar{\Delta}^2$. So, this integral in the deep IR region vanishes as ϵ^2 , i.e., $T_2'^{(7)}(q) \sim \epsilon^2$ (compare with Eq. (8.12)). Thus, in the integrals which contain only the PT parts of the full gluon propagators, the integration is to be effectively taken from some finite value to infinity (for all the loop variables), in accordance with the decomposition (2.24). Let us emphasize once more that the PT parts of the corresponding integrals, however, are not important for us. In accordance with our method they have to be subtracted, anyway.

The next two-gluon contribution into the gluon self-energy is given in Eq. (2.16), so its PT part is a sum of the fifteen terms

$$T_2^{PT}(q) = \sum_{n=1}^{15} T_2^{(n)}(q), \tag{A22}$$

where

$$\begin{aligned}
T_2^{(1)}(q) &= g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) \\
&D^{INP}(q_1) D^{INP}(-q_2) D^{INP}(q - q_1 + q_2) D^{PT}(q - q_1), \tag{A23}
\end{aligned}$$

$$\begin{aligned}
T_2^{(2)}(q) &= g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) \\
&D^{INP}(q_1) D^{INP}(-q_2) D^{PT}(q - q_1 + q_2) D^{INP}(q - q_1), \tag{A24}
\end{aligned}$$

$$\begin{aligned}
T_2^{(3)}(q) &= g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) \\
&D^{INP}(q_1) D^{INP}(-q_2) D^{PT}(q - q_1 + q_2) D^{PT}(q - q_1), \tag{A25}
\end{aligned}$$

$$\begin{aligned}
T_2^{(4)}(q) &= g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) \\
&D^{INP}(q_1) D^{PT}(-q_2) D^{INP}(q - q_1 + q_2) D^{INP}(q - q_1), \tag{A26}
\end{aligned}$$

$$\begin{aligned}
T_2^{(5)}(q) &= g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) \\
&D^{INP}(q_1) D^{PT}(-q_2) D^{INP}(q - q_1 + q_2) D^{PT}(q - q_1), \tag{A27}
\end{aligned}$$

$$T_2^{(6)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{INP}(q_1) D^{PT}(-q_2) D^{PT}(q - q_1 + q_2) D^{INP}(q - q_1), \quad (\text{A28})$$

$$T_2^{(7)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{INP}(q_1) D^{PT}(-q_2) D^{PT}(q - q_1 + q_2) D^{PT}(q - q_1), \quad (\text{A29})$$

$$T_2^{(8)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{INP}(-q_2) D^{INP}(q - q_1 + q_2) D^{INP}(q - q_1), \quad (\text{A30})$$

$$T_2^{(9)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{INP}(-q_2) D^{INP}(q - q_1 + q_2) D^{PT}(q - q_1), \quad (\text{A31})$$

$$T_2^{(10)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{INP}(-q_2) D^{PT}(q - q_1 + q_2) D^{INP}(q - q_1), \quad (\text{A32})$$

$$T_2^{(11)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{INP}(-q_2) D^{PT}(q - q_1 + q_2) D^{PT}(q - q_1), \quad (\text{A33})$$

$$T_2^{(12)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(-q_2) D^{INP}(q - q_1 + q_2) D^{INP}(q - q_1), \quad (\text{A34})$$

$$T_2^{(13)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(-q_2) D^{INP}(q - q_1 + q_2) D^{PT}(q - q_1), \quad (\text{A35})$$

$$T_2^{(14)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(-q_2) D^{PT}(q - q_1 + q_2) D^{INP}(q - q_1), \quad (\text{A36})$$

$$T_2^{(15)}(q) = g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(-q_2) D^{PT}(q - q_1 + q_2) D^{PT}(q - q_1). \quad (\text{A37})$$

Integrating over the gluon momenta q_1 and $-q_2$ in Eqs. (A23), (A24) and (A25), one obtains

$$T_2^{(1)}(q) = \bar{\Delta}_2^4 T_4^0 T_3(0, q, -q) T_3(-q, 0, q) D^{INP}(q) D^{PT}(q) \sim \epsilon, \quad \epsilon \rightarrow 0^+, \quad (\text{A38})$$

$$T_2^{(2)}(q) = \bar{\Delta}_2^4 T_4^0 T_3(0, q, -q) T_3(-q, 0, q) D^{PT}(q) D^{INP}(q) \sim \epsilon, \quad \epsilon \rightarrow 0^+. \quad (\text{A39})$$

Here and below such kind of contributions vanish because of Eq. (7.5), since there is no explicit integration over the gluon momentum q .

$$T_2^{(3)}(q) = \bar{\Delta}_2^4 T_4^0 T_3(0, q, -q) T_3(-q, 0, q) D^{PT}(q) D^{PT}(q). \quad (\text{A40})$$

Integrating first over the gluon momentum q_1 and then over the gluon momentum $q_2 = -q$ in Eq. (A26), one gets

$$T_2^{(4)}(q) = \bar{\Delta}_2^4 T_4^0 T_3(q, 0, -q) T_3(-q, 0, q) D^{PT}(q) D^{INP}(q) \sim \epsilon, \quad \epsilon \rightarrow 0^+. \quad (\text{A41})$$

Integrating first over the gluon momentum q_1 and then over the gluon momentum $q_2 = -q$ in Eq. (A27), one arrives at

$$T_2^{(5)}(q) = \bar{\Delta}_2^4 T_4^0 T_3(q, 0, -q) T_3(-q, 0, q) D^{PT}(q) D^{PT}(q). \quad (\text{A42})$$

Integrating over the gluon momentum q_1 in Eq. (A28), leads to

$$T_2^{(6)}(q) = -\bar{\Delta}_2^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q + q_2, -q) T_3(-q, 0, q) D^{PT}(-q_2) D^{PT}(q + q_2) D^{INP}(q) \sim \epsilon, \quad \epsilon \rightarrow 0^+. \quad (\text{A43})$$

Integrating over the gluon momentum q_1 in Eq. (A29), one obtains

$$T_2^{(7)}(q) = -\bar{\Delta}_2^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q + q_2, -q) T_3(-q, 0, q) D^{PT}(-q_2) D^{PT}(q + q_2) D^{PT}(q). \quad (\text{A44})$$

Integrating over the gluon momentum $-q_2$ in Eq. (A30), one gets

$$T_2^{(8)}(q) = -\bar{\Delta}_2^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 T_3(0, q - q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{INP}(q - q_1) D^{INP}(q - q_1) \sim \epsilon, \quad \epsilon \rightarrow 0^+. \quad (\text{A45})$$

In connection with this integral, let us note that its behavior with respect to ϵ comes from $D^{INP}(q - q_1) D^{INP}(q - q_1) = \Delta^2[(q - q_1)^2]^{-2} \Delta^2[(q - q_1)^2]^{-2} = \epsilon^2 \bar{\Delta}_2^4[(q - q_1)^2]^{-4}$. However, the singularity in the integrand function is not so severe, since both three-gluon vertices depend on the momentum $q - q_1$ linearly. Due to symmetric integration, in fact, the singularity is $\sim [(q - q_1)^2]^{-3}$, which in terms of ϵ produces $1/\epsilon$ pole with residue being the first derivative of the δ -function (see Eq. (6.1)). If one ignores the structure of the corresponding three-gluon vertices, nevertheless, the singularity $\sim [(q - q_1)^2]^{-4}$ produces $1/\epsilon$ pole as well. So, the integral (A45) vanishes as ϵ . Let us note that this and other integrals above and below reproduce the case when the number of loop integrations does not coincide with the number of the gluon propagators. The derivatives of the δ -function should appear, and not the product of the δ -functions at the same point, which has no mathematical meaning, as underlined above.

Integrating over the gluon momentum $-q_2$ in Eq. (A31), one obtains

$$T_2^{(9)}(q) = -\bar{\Delta}_2^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 T_3(0, q - q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{INP}(q - q_1) D^{PT}(q - q_1) = 0. \quad (\text{A46})$$

In connection with this integral, let us note that we cannot integrate directly over the gluon momentum $q_1 = q$, since dependence on it appears in both gluon propagators, namely, $D^{INP}(q - q_1) D^{PT}(q - q_1) = \Delta^2[(q - q_1)^2]^{-2} [(q - q_1)^2]^{-1} = \epsilon \bar{\Delta}_2^2[(q - q_1)^2]^{-3}$. However, the singularity in the integrand function is not so severe, since both three-gluon vertices

depend on the momentum $q - q_1$ linearly. Due to symmetric integration, in fact, the singularity is $\sim [(q - q_1)^2]^{-2}$, which in terms of ϵ produces $1/\epsilon$ pole, which residue is $\delta^4(q - q_1)$. Since $T_3(0, 0, 0) = 0$, one obtains the above-displayed result.

Integrating over the gluon momentum $-q_2$ in Eq. (A32), one arrives at

$$T_2^{(10)}(q) = -\bar{\Delta}_2^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 T_3(0, q - q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(q - q_1) D^{INP}(q - q_1) = 0 \quad (\text{A47})$$

because of the same reason as the previous integral.

Integrating over the gluon momentum $-q_2$ in Eq. (A33), one obtains

$$T_2^{(11)}(q) = -\bar{\Delta}_2^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 T_3(0, q - q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(q - q_1) D^{PT}(q - q_1). \quad (\text{A48})$$

Similar to the integral (A46), we cannot integrate directly over the gluon momentum $q_1 = q$, since dependence on it appears in both gluon propagators, namely, $D^{PT}(q - q_1) D^{PT}(q - q_1) = [(q - q_1)^2]^{-2}$. However, the singularity in the integrand function is not so severe, since both three-gluon vertices depend on the momentum $q - q_1$ linearly. Due to symmetric integration, in fact, the singularity is $\sim [(q - q_1)^2]^{-1}$, which is not the NP IR singularity at all. So, this integral is finite in the $\epsilon \rightarrow 0^+$ limit.

Integrating over the gluon momentum $q_1 = q$ in Eq. (A34), one gets

$$T_2^{(12)}(q) = -\bar{\Delta}_2^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q_2, 0) T_3(-q, q, 0) D^{PT}(q) D^{PT}(-q_2) D^{INP}(q_2) = 0 \quad (\text{A49})$$

Again we cannot integrate directly over the gluon momentum q_2 , since dependence on it appears in both gluon propagators, namely, $D^{INP}(q_2) D^{PT}(-q_2) = \Delta^2 [(q_2)^2]^{-2} [(q_2)^2]^{-1} = \epsilon \bar{\Delta}_2^2 [(q_2)^2]^{-3}$. However, the singularity in the integrand function is not so severe, since the three-gluon vertex depends on the momentum q_2 linearly. Due to symmetric integration, in fact, the singularity is $\sim [(q_2)^2]^{-2}$, which in terms of ϵ produces $1/\epsilon$ pole, which residue is $\delta^4(q_2)$. Since $T_3(0, 0, 0) = 0$, one obtains the above-displayed result.

Integrating over gluon momentum $q_1 = q + q_2$ in Eq. (A35), one obtains

$$T_2^{(13)}(q) = -\bar{\Delta}_2^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, 0, q_2) T_3(-q, q + q_2, -q_2) D^{PT}(q + q_2) D^{PT}(-q_2) D^{PT}(-q_2). \quad (\text{A50})$$

We cannot integrate directly over the gluon momentum $-q_2$, since dependence on it appears in both gluon propagators, namely, $D^{PT}(-q_2) D^{PT}(-q_2) = [(q_2)^2]^{-2}$. However, the singularity in the integrand function is not so severe, since the three-gluon vertex depends on the momentum $-q_2$ linearly. Due to symmetric integration, in fact, the singularity is $\sim [-q_2]^2]^{-1}$, which is not the NP IR singularity at all. So, this integral is finite in the $\epsilon \rightarrow 0^+$ limit.

Integrating over the gluon momentum $q_1 = q$ in Eq. (A36), we get

$$T_2^{(14)}(q) = \bar{\Delta}_2^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q_2, 0) T_3(-q, q, 0) D^{PT}(q) D^{PT}(-q_2) D^{PT}(q_2), \quad (\text{A51})$$

and it is finite in the $\epsilon \rightarrow 0^+$ limit in the same way as the previous integral. The last integral (A37) for $T_2^{(15)}(q)$ remains the same.

Summing up nonzero contributions, one finally obtains

$$\begin{aligned}
T_2(q) = & \bar{\Delta}_2^4 T_4^0 T_3(0, q, -q) T_3(-q, 0, q) D^{PT}(q) D^{PT}(q) \\
& + \bar{\Delta}_2^4 T_4^0 T_3(q, 0, -q) T_3(-q, 0, q) D^{PT}(q) D^{PT}(q) \\
& - \bar{\Delta}_2^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q + q_2, -q) T_3(-q, 0, q) D^{PT}(-q_2) D^{PT}(q + q_2) D^{PT}(q) \\
& - \bar{\Delta}_2^2 \int \frac{id^4 q_1}{(2\pi)^4} T_4^0 T_3(0, q - q_1, q_1 - q) T_3(-q, q_1, q - q_1) D^{PT}(q_1) D^{PT}(q - q_1) D^{PT}(q - q_1) \\
& - \bar{\Delta}_2^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, 0, q_2) T_3(-q, q + q_2, -q_2) D^{PT}(q + q_2) D^{PT}(-q_2) D^{PT}(-q_2) \\
& - \bar{\Delta}_2^2 \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q_2, 0) T_3(-q, q, 0) D^{PT}(q) D^{PT}(-q_2) D^{PT}(q_2) \\
& + g^4 \int \frac{id^4 q_1}{(2\pi)^4} \int \frac{id^4 q_2}{(2\pi)^4} T_4^0 T_3(-q_2, q - q_1 + q_2, q_1 - q) T_3(-q, q_1, q - q_1) \\
& \times D^{PT}(q_1) D^{PT}(-q_2) D^{PT}(q - q_1 + q_2) D^{PT}(q - q_1).
\end{aligned} \tag{A52}$$

Again one can conclude that two-loop skeleton contribution into the gluon self-energy due to four- and three-gluon vertices does not produce explicitly any problems in the $\epsilon \rightarrow 0^+$ limit.

The PT part of nonlinear pure gluon part is a sum of four terms, namely

$$T_g^{PT}[D](q) = \frac{1}{2} T_t^{PT} + \frac{1}{2} T_1^{PT}(q) + \frac{1}{2} T_2^{PT}(q) + \frac{1}{6} T_2'^{PT}(q), \tag{A53}$$

where each term is given by Eqs. (A1), (A6), (A21) and (A52). The total sum $T_g^{PT}[D](q)$ is finite in the $\epsilon \rightarrow 0^+$ limit. This ends the investigation of the gluon SD equation (8.20) for its PT part. The main goal of this appendix to show that the PT part does not explicitly produce any problems in the $\epsilon \rightarrow 0^+$ limit has been achieved, indeed.

-
- [1] W. Marciano, H. Pagels, Phys. Rep. C 36 (1978) 137.
 - [2] G. 't Hooft, hep-th/0408183; A.M. Polyakov, hep-th/0407209.
 - [3] M. Baker, C. Lee, Phys. Rev. D 15 (1977) 2201;
U. Bar-Gadda, Nucl. Phys. B 163 (1980) 312.
 - [4] E.G. Eichten, F.L. Feinberg, Phys. Rev. D 10 (1974) 3254.
 - [5] Behavior of the Solutions to the Bethe-Salpeter Equation (Ed. N. Nakanishi), Prog. Theor. Phys. Suppl. 95 (1988);
K. Ladanyi, Ann. Phys. (N.Y.), 130 (1980) 427.
 - [6] J. C. Taylor, Nucl. Phys. B 33 (1971) 436;
A. A. Slavnov, Sov. Jour. Theor. Math. Phys. 10 (1972) 153.
 - [7] G. 't Hooft, Nucl. Phys. B 33 (1971) 173;
S.K. Kim, M. Baker, Nucl. Phys. B 164 (1980) 152;
S.-H.H. Tye, E. Tomboulis, E.C. Poggio, Phys. Rev. D 11 (1975) 2839.
 - [8] P. Pascual, R. Tarrach, Nucl. Phys. B 174 (1980) 123;
B.W. Lee, Phys. Rev. D 9 (1974) 933.
 - [9] H. Pagels, Phys. Rev. D 15 (1977) 2991.
 - [10] S. Mandelstam, Phys. Rev. D 20 (1979) 3223.
 - [11] A. Hadicke, Intr. Jour. Mod. Phys. A 6 (1991) 3321;
C.D. Roberts, A.G. Williams, Prog. Part. Nucl. Phys. 33 (1994) 477.
 - [12] V. Gogohia, Gy. Kluge, M. Prisznyak, hep-ph/9509427.
 - [13] T. Maskawa, N. Nakajima, Prog. Theor. Phys. 52 (1974) 1326.
 - [14] S.L. Adler, Prog. Theor. Phys. Suppl. 86 (1986) 12.
 - [15] M. Bando, M. Harada, T. Kugo, Prog. Theor. Phys. 91 (1994) 927.
 - [16] V. Gogohia, Phys. Lett. B 468 (1999) 279, hep-ph/9908302;
V. Gogohia, hep-th/0406064.
 - [17] M.E. Peskin, D.V. Schroeder, An Introduction to Quantum Field Theory (AW, Advanced Book Program, 1995).
 - [18] C. Itzykson, J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980).
 - [19] K.G. Wilson, Phys. Rep. C 23 (1976) 331.
 - [20] V. Gogohia, Phys. Lett. B 531 (2002) 321, see also hep-ph/0104296.

- [21] V. Gogohia, Gy. Kluge, Phys. Rev. D 66 (2002) 056013, hep-ph/0104296;
V. Gogohia, Gy. Kluge, I. Vargas de Usera, Phys. Lett. B 576 (2003) 243, hep-ph/0310253;
V. Gogohia, Gy. Kluge, J. Nyiri, hep-ph/0204347.
- [22] G. 't Hooft, Nucl. Phys. B 75 (1974) 461.
- [23] M. Faber, A.N. Ivanov, Phys. Lett. B 563 (2003) 231;
M. Faber, A.N. Ivanov, J. Phys. A: Math. Gen. 36 (2003) 7837.
- [24] I.M. Gel'fand, G.E. Shilov, Generalized Functions, (Academic Press, New York, 1968), Vol. I.
- [25] G. 't Hooft, M. Veltman, Nucl. Phys. B 44 (1972) 189;
C.G. Bollini, J.J. Giambiagi, Nuovo Cim. B 12 (1972) 20;
J.F. Ashmore, Lett. Nuovo Cim. 4 (1972) 289.
- [26] J.E. Mandula, Phys. Rep. 315 (1999) 273.
- [27] G. 't Hooft, Conference on Lagrangian Field Theory, Marseille, 1972;
D.J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343;
H.D. Politzer, Phys. Rev. Lett. 30 (1973) 1346.
- [28] V. Gogohia, Phys. Lett. B 485 (2000) 162, hep-ph/0006063.
- [29] S.G. Matynian, S.K. Savvidy, Nucl. Phys. B 134 (1978) 539.
- [30] F. Wilczek, Proc. of Inter. Conf., QCD - 20 Years Later, Aachen, June 9-13, 1992, v. 1.
- [31] R.D. Field, Application of Perturbative QCD (Addison-Wesley, 1990);
T. Muta, Foundation of QCD (World Scientific, Singapore, 1987).
- [32] D.J. Gross, F. Wilczek, Phys. Rev. D 8 (1973) 3633.
- [33] J. Kogut, L. Susskind, Phys. Rev. D 9 (1974) 3501;
L. Susskind, J. Kogut, Phys. Rep. C 23 (1976) 348.
- [34] H. Fritzsch, M. Gell-Mann, H. Leutwyler, Phys. Lett. B 47 (1973) 365;
S. Weinberg, Phys. Rev. Lett. 31 (1973) 494.
- [35] H. Georgi, S. Glashow, Phys. Rev. Lett. 32 (1974) 438.
- [36] J.L. Gervais, A. Neveu, Phys. Rep. C 23 (1976) 240;
T. Banks, A. Casher, Nucl. Phys. B 169 (1980) 103.
- [37] S. Coleman, E. Weinberg, Phys. Rev. D 7 (1973) 1888;
D.J. Gross, A. Neveu, Phys. Rev. D 10 (1974) 3235.
- [38] L. von Smekal, A. Hauck, R. Alkofer, Annals Phys. 267 (1998) 1.
- [39] V.N. Gribov, Eur. Phys. J. C 10 (1999) 71;
V.N. Gribov, Gauge Theories and Quark Confinement (PHASIS, Moscow, 2002).
- [40] J.S. Ball, T.-W. Chiu, Phys. Rev. D 22 (1980) 2542, 2550.
- [41] G. Leibbrandt, Noncovariant Gauges (WS, Singapore, 1994);
A. Bassetto, G. Nardelli, R. Soldati, Yang-Mills Theories in Algebraic Non Covariant Gauges (WS, Singapore, 1991).
- [42] Confinement, Duality, and Nonperturbative Aspects of QCD, edited by P. van Baal, NATO ASI Series B: Physics, vol. 368 (Plenum, New York, 1997).
- [43] Non-Perturbative QCD, Structure of the QCD Vacuum, edited by K-I. Aoki, O. Miyamura, and T. Suzuki [Prog. Theor. Phys. Suppl. 131 (1998) 1].
- [44] Mark D. Roberts, hep-th/0012062.
- [45] P. van Baal, Nucl. Phys. B (Proc. Suppl.) 63A-C (1998) 126;
F. Lenz, J.W. Negele, M. Thies, Phys. Rev. D 69 (2004) 074009;
G. Gabadadze, M. Shifman, hep-ph/0206123.
- [46] N.K. Nielsen, P. Olesen, Nucl. Phys. B 144 (1978) 376.
- [47] L. Del Debbio, M. Faber, J. Giedt, J. Greensite, S. Olejnik, Phys. Rev. D 58 (1998) 094501;
M. Engelhardt, H. Reinhardt, Nucl. Phys. B 585 (2000) 591.
- [48] T. Schafer, E.V. Shuyrak, Rev. Mod. Phys. 70 (1998) 323.
- [49] V. A. Rubakov, Classical Theory of Gauge Fields, (Princeton University Press, 2002).
- [50] V. Gogohia, Gy. Kluge, Phys. Rev. D 62 (2000) 076008, hep-ph/0002003.
- [51] V. Gogohia, Gy. Kluge, H. Toki, T. Sakai, Phys. Lett. B 453 (1999) 281, hep-ph/9810516.
- [52] T. Kugo, I. Ojima, Prog. Theor. Phys. Suppl. 66 (1979) 1.
- [53] K. Nishijima, Prog. Theor. Phys. 75 (1986) 1221;
K. Nishijima, Czech J. Phys. 46 (1996) 1;
M. Chaichian, K. Nishijima, Eur. Phys. Jour. C 22 (2001) 463;
R. Oehme, W. Zimmermann, Phys. Rev. D 21 (1980) 471, 1661.
- [54] K.-I. Kondo, hep-th/0303251.
- [55] M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. B 147 (1979) 385, 448;
V.A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, Nucl. Phys. B 191 (1981) 301.
- [56] V. I. Zakharov, Int. Jour. Mod. Phys. A 14 (1999) 4865.
- [57] I.M. Gel'fand, G.E. Shilov, Generalized Functions, (Academic Press, New York, 1968), Vol. II.
- [58] M. Baker, J. S. Ball, F. Zachariasen, Nucl. Phys. B 186 (1981) 531, 560.
- [59] N. Brown, M. R. Pennington, Phys. Rev. D 39 (1989) 2723.
- [60] D. Atkinson et al., J. Math. Phys. 25 (1984) 2095.
- [61] L. G. Vachnadze, N. A. Kiknadze, A. A. Khelashvili, Sov. Jour. Teor. Math. Phys. 102 (1995) 47.

- [62] A. I. Alekseev, B. A. Arbuzov, *Mod. Phys. Lett. A* 13 (1998) 1747;
X.-H. Guo, T. Huang, *Il Nuovo Cim. A* 110 (1997) 799;
T.H. Phat, N.T. Anh, *Il Nuovo Cim. A* 110 (1997) 337.
- [63] F. Strocchi, *Phys. Lett. B* 62 (1976) 60.
- [64] K. B. Wilson, *Phys. Rev. D* 10 (1974) 2445;
M. Bander, *Phys. Rep.* 75 (1981) 205;
G. B. West, *Phys. Lett. B* 115 (1982) 468.
- [65] K. D. Born et al., *Phys. Lett. B* 329 (1994) 325;
V. M. Miller et al., *Phys. Lett. B* 335 (1994) 71.
- [66] C. Michael, *Nucl. Phys. B (Proc. Suppl.)* 42 (1995) 147;
G. Damm, W. Kerler, V. K. Mitjushkin, *Phys. Lett. B* 443 (1998) 88;
G. Burgio, F. Di Renzo, C. Parrinello, C. Pittori, *Nucl. Phys. B (Proc. Suppl.)* 73 (1999) 623;
G. Burgio, F. Di Renzo, G. Marchesini, E. Onofri, *Nucl. Phys. B (Proc. Suppl.)* 63A-C (1998) 805.
- [67] M. Lusher et al., *Nucl. Phys. B* 413 (1994) 481.
- [68] F.N. Ndili, *hep-ph/0107195*;
A.V. Nesterenko, *Phys. Rev. D* 64 (2001) 116009.
- [69] V.P. Nair, C. Rosenzweig, *Phys. Rev. D* 31 (1985) 401;
G. Ripka, *Lectures Notes in Physics (LNP)*, Springer, Verlag, v. 639, 2004, *hep-ph/0310102*.
- [70] H. Narnhofer, W. Thirring, in *Springer Tracks Mod. Phys.* 119 (1990) 1.
- [71] K. Johnson, L. Lellouch, J. Polonyi, *Nucl. Phys. B* 367 (1991) 675.
- [72] U. Ellwanger, M. Hirsch, A. Weber, *Eur. Phys. Jour. C* 1 (1998) 563;
D.-U. Jingnickel, C. Wetterich, *hep-ph/0710397*.
- [73] V.Sh. Gogohia, *Inter. Jour. Mod. Phys. A* 9 (1994) 759.
- [74] I. Halperin, A. Zhitnitsky, *Nucl. Phys. B* 539 (1999) 166.
- [75] E. Witten, *Nucl. Phys. B* 156 (1979) 269;
G. Veneziano, *Nucl. Phys. B* 159 (1979) 213.
- [76] V. Gogohia, *Phys. Lett. B* 501 (2001) 60, *hep-ph/0005302*.
- [77] M. Gell-Mann, R.J. Oakes, B. Renner, *Phys. Rev.* 175 (1968) 2195.
- [78] J.M. Cornwall, R. Jackiw, E. Tomboulis, *Phys. Rev. D* 10 (1974) 2428.
- [79] A. Barducci et al., *Phys. Rev. D* 38 (1988) 238;
R.W. Haymaker, *La Riv. del Nuovo Cim.*, 8 (1991) 1.
- [80] P. Castorina, S.-Y. Pi, *Phys. Rev. D* 31 (1985) 411;
V.P. Gusynin, Yu.A. Sitenko, *Z. Phys. C* 29 (1985) 547.
- [81] V.N. Gribov, *Nucl. Phys. B* 139 (1978) 1.
- [82] Y.L. Dokshitzer, D.E. Kharzeev, *hep-ph/0404216*.
- [83] M. Creutz, *Quarks, Gluons and Lattice* (Cambridge, 1983);
A.S. Kronfeld, *hep-ph/0209321*.
- [84] P. Hasenfratz, *Prog. Theor. Phys. Suppl.* 131 (1998) 189.
- [85] M.-C. Chu, J.M. Grandy, S. Huang, J.W. Negele, *Phys. Rev. D* 49 (1994) 6039.
- [86] T. DeGrand, A. Hasenfratz, T.G. Kovacs, *Nucl. Phys. B* 505 (1997) 417.
- [87] A.M. Polyakov, *Nucl. Phys. B* 486 (1997) 23.
- [88] A.I. Alekseev, B.A. Arbuzov, *hep-ph/0407056*.
- [89] M.N. Chernodub, M.I. Polikarpov, V.I. Zakharov, *hep-ph/9903272*.
- [90] V. Gogohia, Gy. Kluge, *Phys. Lett. B* 477 (2000) 387;
V. Gogohia, M. Priznyak, *Phys. Lett. B* 494 (2000) 109.
- [91] A. Jaffe, E. Witten, *Quantum Yang-Mills Theory*, <http://www.claymath.org/prize-problems/>.
- [92] V. Gogohia, in: *Proc. of the Sixth Workshop on Non-Perturbative QCD*, eds. H. M. Fried, Y. Gabellini, B. Muller (WS, Singapore, 2002) p. 297, *hep-ph/0109196*.
- [93] R. Feynman, *Nucl. Phys. B* 188 (1981) 479.
- [94] V. Gogohia, H. Toki, T. Sakai, Gy. Kluge, *Inter. Jour. Mod. Phys. A* 15 (2000) 45, see also *hep-ph/9810516*.
- [95] N.N. Bogoliubov, D.V. Shirkov, *Introduction to the Theory of Quantised Fields*, Interscience Publisher (NY, 1959).
- [96] E.R. Speer, *Jour. Math. Phys.* 15 (1974) 1.
- [97] V. Gogohia, *Phys. Lett. B* 584 (2004) 225, *hep-ph/0304163*.
- [98] F. Wilczek, *hep-ph/0201222*.
- [99] D. Zwanziger, *Nucl. Phys. B* 364 (1991) 127.
- [100] P.J. Silva, O. Oliveira, *Nucl. Phys. B* 690 (2004) 177.
- [101] C. Preparata, *Phys. Rev. D* 7 (1973) 2973.
- [102] R. Fukuda, T. Kugo, *Nucl. Phys. B* 117 (1976) 250.
- [103] G. Pocsik, T. Torma, *Acta. Phys. Hun.* 62 (1987) 101, 107;
Z. Fodor, *Acta Phys. Pol. B* 19 (1988) 21.
- [104] V.Sh. Gogokhia, *Phys. Rev. D* 40 (1989) 4157;
V.Sh. Gogokhia, G.V. Efimov, B.A. Magradze, *Preprint JINR P2-88-127*.
- [105] V. Gogohia, *hep-ph/0404054*.
- [106] B. Magradze, *AMS/IP Studies in Adv. Math.* 13 (1999) 399.